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Supersymmetric Quantum Field Theory in Zero Dimension

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ABSTRACT: The aim of this article is to review some aspects of supersymmetric quantum field theory in zero dimension. The basic construction of a generic quantum field theory is discussed in terms of its domain manifold and field contents. Then, we introduce the path integral formalism for quantum field theories in zero dimension. With a special choice of the action, the partition fucntion is shown to have an additional symmetry, called the supersymmetry. For the supersymmetric theory, we show that the partition function is localised on the loci where the fermionic supersymmetry transformation vanishes. It is also shown that the superpotential is invariant under deformations.

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1 Introduction

A quantum field theory (QFT) is a mathematical framework developed by physicists based on a few guiding principles from special relativity and quantum mechanics. Most of the constructions for a QFT are mathematical conjectures and few QFTs have been proven to exist with mathematical rigour. However, one particular QFT, namely the Standard Model, has shown to be a great success, consistent with experimental results up to an unprecedented level of accuracy. In modern theoretical physics, it is generally believed that all physics should be based on some QFTs. In this review, we will discuss QFTs in zero dimension. Studying this simplistic model demonstrates some of the powerful techniques applicable to higher dimensional cases. This review follows Chapter 9 of *Mirror Symmetry* [1].

1.1 Construction of Quantum Field Theory

The base structure of a QFT is a metric manifold $\{M, g\}$ of dimension n. The main objects of study are called *fields*, which are to be integrated over the manifold M in the so-called *path integral formalism*. These fields are therefore functions or tensor bundles over M. For example, a *quantum gauge theory* considers gauge fields, i.e., principal bundles with connections, and matter field, i.e., vector bundles. In this formalism, to study a quantum field theory means to compute all the path integrals of interest.

We can add more structures into mix. For example, a *sigma model* considers the maps

$$X : M \mapsto N \tag{1.1}$$

for some target manifold N. As another example, integrating over choices of metric on the manifold M gives *quantum gravity*.

In the case where the manifold M has some boundaries

$$\partial M = \cup_i B_i \,. \tag{1.2}$$

The field configurations on B_i, i.e., boundary conditions, form a Hilbert space $\mathcal{H}_i$. The path integral can be viewed as linear maps

$$\oplus_i \mathcal{H}_i \to \mathbb{C} \,. \tag{1.3}$$

In addition to the path integral formalism, there is an alternative approach called the *operator formalism*, in which we promote fields to operators on the Hilbert space obeying some canonical quantisation relation.

2 Zero-Dimensional Quantum Field Theory

For a zero-dimensional QFT, the domain manifold M is simply a point. The operator formalism does not exist here since a point does not have boundaries.

Let's consider the simplest field, a real scalar function on M, i.e., $X : M \to \mathbb{R}$. The path integral formalism starts by defining the partition function

$$Z := \int dX\, e^{-S(X)} \,, \tag{2.1}$$

where $S(X)$ is a function of X called the action, which specifies the system at the classical level.[1] In studying a QFT, we are also interested in computing path integrals of functions of fields with $\exp(-S)$ as the weight, called correlation functions given by

$$\langle f(X) \rangle := \int dX\, f(X) e^{-S(X)} \,. \tag{2.2}$$

[1]In higher dimensions, $S[X]$ is a functional of the function X.

It is conventional to normalise the correlation functions by a factor of the (free) partition function. In practice, the computation of correlation functions rely on deforming the action by a "source" term

$$\delta S := -\sum_i a_i f_i(X)\,, \tag{2.3}$$

and taking derivatives with respect to the constants a_i.

$$\langle f_i(X)\rangle = \frac{\partial}{\partial a_i}\int dX\, e^{-S-\delta S}\bigg|_{a_i=0}\,. \tag{2.4}$$

As an example, consider a toy model with action

$$S(X) = \frac{1}{2}\alpha X^2 + i\epsilon X^3 \tag{2.5}$$

for some constants α and ϵ. The (normalised) partition function can be written as

$$Z(\alpha,\epsilon) = \int dX \sum_{n=0}^{\infty} e^{-\frac{1}{2}\alpha X^2}\frac{(-i\epsilon X^3)^n}{n!}\,. \tag{2.6}$$

Each term in the sum can be evaluated using the generating function

$$f(\alpha, J) = \int dX\, e^{-\frac{1}{2}\alpha X^2 + JX} = \left(\frac{\alpha}{2\pi}\right)^{-\frac{1}{2}} e^{\frac{J^2}{2\alpha}} \tag{2.7}$$

via

$$\int dX\, X^r e^{-\frac{1}{2}\alpha X^2} = \frac{\partial^r}{\partial J^r} f(\alpha, J)\bigg|_{J=0}\,. \tag{2.8}$$

This computational device gives rise to the famous "Feynman diagrams", which we shall not discuss in details here.

The above construction can also be generalised easily to multi-variable cases using the integral formula

$$Z(M) = \int\left(\prod_k dX^k\right) e^{-\frac{1}{2}X^i M_{ij} X^j} = \det\left(\frac{M}{2\pi}\right)^{-\frac{1}{2}} \tag{2.9}$$

where M is a positive-definite and invertible matrix.[2]

2.1 Fermions and Supersymmetry

To describe fermions in the path integral language, we need to introduce Grassmann "variables" $\{\psi_i \mid i = 1, \ldots, N\}$, which form the basis of an exterior algebra.[3] Note that these are not variables in the usual sense. There is a $\mathbb{Z}_2$ grading that assigns 0 to bosonic variables and 1 to fermionic variables. The bosonic and fermionic variables are also said to be Grassmann even and odd respectively. The commutator is generalised to a graded commutator $[\cdot,\cdot\}$, defined by

$$[A, B\} = AB - (-1)^{ab} BA\,, \tag{2.10}$$

where a, b are the gradings of A, B. This reduces to a commutator $[\cdot,\cdot]$ when either variable is bosonic, and becomes an anti-commutator $\{\cdot,\cdot\}$ when both variables are fermionic. Note that Grassman variables are nilpotent, i.e.,

$$\psi\psi = 0\,. \tag{2.11}$$

[2]The Einstein summation convention has been used for repeated indices.
[3]Not to be confused with superspace coordinates.

To integrate over fermions, we need to define Berezin "integration" by

$$\int d\psi = 0\,, \tag{2.12}$$

$$\int d\psi\, \psi = 1\,, \tag{2.13}$$

$$\int d\psi_1 \cdots d\psi_N\, \psi_N \cdots \psi_1 = 1\,. \tag{2.14}$$

Note that the sign in Equation 2.14 is a choice of convention.

Now an action describing both bosons and fermions is a function of both X and ψ. Since the original action $S(X)$ is Grassmann even, to retain the original bosonic piece the new action $S(X, \psi)$ must also be Grassmann even. Hence there must be an even number of fermionic variables.

Consider an action involving only fermionic variables ψ_a

$$S = \frac{1}{2}\psi_a M_{ab}\psi_b\,, \tag{2.15}$$

The partition function is given by

$$Z = \int \prod_k d\psi_k\, e^{-\frac{1}{2}\psi_a M_{ab}\psi_b} = \mathrm{Pf}(M)\,, \tag{2.16}$$

where $\mathrm{Pf}(M)$ is the pfaffian of M and $\mathrm{Pf}(M)^2 = \det(M)$. Note that the Berezin gaussian integral gives a factor of $\det(M)^{1/2}$, while the normal gaussian integral produces $\det(M)^{-1/2}$.

The partition function of a general action $S[X, \psi]$ is given by

$$Z = \int \prod_i dX^i \prod_a d\psi_a\, e^{-S[X,\psi]}\,, \tag{2.17}$$

where the indices of X and ψ have been depreciated in the arguments of the action.

The simplest non-trivial example involves one bosonic variable and two fermionic variables. The most general action in this case is given by

$$S(X, \psi_1, \psi_2) = S_0(X) - \psi_1\psi_2 S_1(X). \tag{2.18}$$

The partition function is

$$\begin{aligned} Z &= \int dX d\psi_1 d\psi_2\, e^{-S_0 + \psi_1\psi_2 S_1(X)} && (2.19)\\ &= \int dX d\psi_1 d\psi_2\, e^{-S_0}(1 + \psi_1\psi_2 S_1(X)) && (2.20)\\ &= \int dX d\psi_1 d\psi_2\, e^{-S_0(X)} + \int dX d\psi_1 d\psi_2\, e^{-S_0(X)}\psi_1\psi_2 S_1(X) && (2.21)\\ &= \int dX e^{-S_0} S_1(X)\,. && (2.22) \end{aligned}$$

For a special choice of $S_0(X)$ and $S_1(X)$, the theory gains a continuous symmetry, called the *supersymmetry*, which mixes bosonic and fermionic variables. In this special choice, the action is

$$S(X, \psi_1, \psi_2) := \frac{1}{2}(h'(X))^2 - h''(X)\psi_1\psi_2\,, \tag{2.23}$$

where $h(X)$ is a real function of X, and h' denotes its derivative with respect to X. The (infinitesimal) supersymmetry transformation is given by

$$\delta X = \epsilon^1 \psi_1 + \epsilon^2 \psi_2 \,, \tag{2.24}$$

$$\delta \psi_1 = \epsilon^2 h'(X) \,, \tag{2.25}$$

$$\delta \psi_2 = -\epsilon^1 h'(X) \,, \tag{2.26}$$

where ϵ^1 and ϵ^2 are (infinitesimal) fermionic parameters.

It is easy to show that the change of the action

$$\delta S = \delta X \frac{\partial S}{\partial X} + \delta \psi_a \frac{\partial S}{\partial \psi_a} \tag{2.27}$$

vanishes under this transformation. However, for the quantum field theory to be invariant, the integration measure in the path integral is also required to be invariant. To compute the variation of the integration measure, we need to introduce the concepts of *supertrace* and *superdeterminant*. Consider a linear map

$$M : V \to V \,, \tag{2.28}$$

where $V = V_0 \otimes V_1$ is a super vector space. Then the map M can be written as a matrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} , \tag{2.29}$$

where $A : V_0 \to V_0$ and $D : V_1 \to V_1$. The supertrace is defined by

$$\operatorname{str} M := \operatorname{tr} A - \operatorname{tr} D, . \tag{2.30}$$

The superdeterminant $\operatorname{sdet} M$ can be defined such that

$$\delta(\log \operatorname{sdet} M) = \operatorname{str}(M^{-1} \delta M) \,, \tag{2.31}$$

$$\operatorname{sdet} \mathbb{1} = 1 \,. \tag{2.32}$$

The supersymmetry transformation on the super vector space

$$\begin{pmatrix} X \\ \psi_1 \\ \psi_2 \end{pmatrix} \mapsto \begin{pmatrix} X + \delta X \\ \psi_1 + \delta\psi_1 \\ \psi_2 + \delta\psi_2 \end{pmatrix} \tag{2.33}$$

gives a jacobian

$$J = \operatorname{sdet}(\mathbb{1} + E) \,, \tag{2.34}$$

where

$$E = \begin{pmatrix} 0 & \epsilon^1 & \epsilon^2 \\ \epsilon^2 h'' & 0 & 0 \\ -\epsilon^1 h'' & 0 & 0 \end{pmatrix} . \tag{2.35}$$

Using the definition of superdeterminant, we have

$$J = \exp \log \operatorname{sdet}(\mathbb{1} + E) \tag{2.36}$$

$$= \exp\big(\log \operatorname{sdet} \mathbb{1} + \operatorname{str}(\mathbb{1}^{-1} E)\big) \tag{2.37}$$

$$= 1 \,. \tag{2.38}$$

We have shown that the integration measure is indeed invariant under the supersymmetry transformation. Therefore, we say this theory has supersymmetry.

2.2 Localisation Principle

One of the striking features of this supersymmetric theory is that the partition function is (almost) always zero. The idea is to use the supersymmetry transformation to set one of the fermionic variables in the action to zero, thus making the partition function zero using the Berezin integration rules.

For example, we can choose

$$\epsilon^1 = \epsilon^2 = -\frac{\psi_1}{h'(X)} \tag{2.39}$$

to set ψ_1 to zero, provided that $h'(X)$ is non-vanishing for all X. The transformed fields are given by

$$\begin{pmatrix} X \\ \psi_1 \\ \psi_2 \end{pmatrix} \to \begin{pmatrix} X - \frac{\psi_1\psi_2}{h'(X)} \\ 0 \\ \psi_2 + \psi_1 \end{pmatrix}. \tag{2.40}$$

However, this change of variables unfortunately gives a vanishing jacobian, which does not help in evaluating the original integral. Instead, we consider a new transformation

$$\begin{pmatrix} X \\ \psi_1 \\ \psi_2 \end{pmatrix} \to \begin{pmatrix} \hat{X} \\ \hat{\psi}_1 \\ \hat{\psi}_2 \end{pmatrix} = \begin{pmatrix} X - \frac{\psi_1\psi_2}{h'(X)} \\ \alpha(X)\psi_1 \\ \psi_2 + \psi_1 \end{pmatrix}, \tag{2.41}$$

where $\alpha(X)$ is a real function of X. To compute the jacobian, we need to use the formula

$$\operatorname{sdet}(X) = \det(A)\det\big(D - CA^{-1}B\big)^{-1} \tag{2.42}$$

for an even super matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where A, D are even and B, C are odd. The jacobian is then computed to be

$$J = \operatorname{sdet}\begin{pmatrix} 1 + \psi_1\psi_2 h''(X)/(h'(X))^2 & -\psi_2/h'(X) & \psi_1/h'(X) \\ \alpha'(X)\psi_1 & \alpha(X) & 0 \\ 0 & 1 & 1 \end{pmatrix} \tag{2.43}$$

$$= \left(\alpha(X) - \frac{\alpha'(X)\psi_1\psi_2}{h'(X)}\right)^{-1}\left(1 + \frac{\psi_1\psi_2 h''(X)}{(h'(X))^2}\right) \tag{2.44}$$

$$= \alpha(\hat{X})^{-1}\left(1 + \frac{\hat{\psi}_1\hat{\psi}_2 h''(\hat{X})}{\alpha(\hat{X})h''(\hat{X})}\right). \tag{2.45}$$

We then apply this change of variables to the integration measure to get

$$dX d\psi_1 d\psi_2 = \left(\alpha(\hat{X}) - \frac{\hat{\psi}_1\hat{\psi}_2 h''(\hat{X})}{[h'(\hat{X})]^2}\right) d\hat{X} d\hat{\psi}_1 d\hat{\psi}_2 \,. \tag{2.46}$$

The action is invariant under the original transformation with $\alpha = 0$ in Equation 2.40, thus

$$S(X, \psi_1, \psi_2) = S(\hat{X}, 0, \hat{\psi}_2)\,. \tag{2.47}$$

Combining Equation 2.46 and Equation 2.47 into the partition function gives

$$Z = \int d\hat{X} d\hat{\psi}_2\, e^{-S(\hat{X},0,\hat{\psi}_2)} \int d\hat{\psi}_1 - \int d\hat{X} d\hat{\psi}_1 d\hat{\psi}_2\, \hat{\psi}_1\hat{\psi}_2 \frac{h''(\hat{X})}{(h'(\hat{X}))^2} e^{-S(\hat{X},0,\hat{\psi}_2)}\,. \tag{2.48}$$

The first term vanishes because $\int d\hat{\psi}_1 = 0$. The second term vanishes since $h''/(h')^2 = -[(h')^{-1}]'$ is a total derivative. Hence we conclude that

$$Z = 0 \quad \text{if} \quad h'(X) \neq 0 \quad \forall \quad X \,. \tag{2.49}$$

Now consider a more general situation where $h'(X)$ may be zero for some X. The partition function only gets contributions from the points where $h'(X) = 0$, i.e., the critical points of $h(X)$. We say that this path integral is localised at the loci where the fermions vanish $\psi_i \mapsto 0$. This is a generic feature for supersymmetric QFTs, under the name of *localisation principle*.

We can Taylor expand the superpotential $h(X)$ around its critical points X_c to get

$$h(X) = h(X_c) + \frac{1}{2}A(X_c)(X - X_c)^2 + \cdots \,. \tag{2.50}$$

Then the localised partition function (normalised to $\sqrt{2\pi}$) is given by

$$Z = \frac{1}{\sqrt{2\pi}} \sum_{X_c} \int dX d\psi_1 d\psi_2 \, e^{-\frac{1}{2}A(X_c)^2(X-X_c)^2 + A(X_c)\psi_1\psi_2} \tag{2.51}$$

$$= \sum_{X_c} \frac{A(X_c)}{|A(X_c)|} \tag{2.52}$$

$$= \sum_{\{X_c | h'(X_c)=0\}} \frac{h''(X_c)}{|h''(X_c)|} \,. \tag{2.53}$$

The result basically says that the partition function is the sum of the signs of $h''(X)$ at the critical points of $h(X)$. If the leading order of h is odd, then $Z = 0$ because there are as many critical points with positive h'' as with negative. If the leading order is even, then $Z = \pm 1$ depending on whether the leading term is positive or negative.

2.3 Deformation Invariance

Now we know that the partition function is largely independent of the details of the superpotential $h(X)$. In fact, the partition function is invariant under (almost) any change in the superpotential, which is called *deformation invariance*.

To prove this, we first need to make the following observation. If we have a QFT with a symmetry, then the correlation functions of quantities that are variations of some fields under the symmetry vanish. Schematically, if the variation of action $\delta S = 0$ and the jacobian $J = 1$ for some transformation δ, then

$$\langle f \rangle = \int f e^{-S} = \int \delta g e^{-S} = \int \delta(g e^{-S}) = 0 \,, \tag{2.54}$$

where f and g are some fields. The validity of this statement depends on the asymptotic behaviour of g. It holds as long as g is not too big at infinity in the field space.

Now consider the change $h(X) \mapsto h(X) + \rho(X)$, the variation of the action is given by

$$\delta_\rho S = h'\rho' - \rho''\psi_1\psi_2 \,. \tag{2.55}$$

This can be written (up to a factor of ϵ) as the supersymmetry transformation of the field $g = \rho'(X)\psi_1$ with parameters $\epsilon^1 = \epsilon^2 = \epsilon$. Hence we can immediately conclude that the correlation function $\langle \delta_\rho S \rangle$ vanishes using the above argument. This implies that the partition function is invariant under the change in the superpotential $h(X)$ since

$$\delta_\rho Z = -\int \delta_\rho S e^{-S} = -\langle \delta_\rho S \rangle \,. \tag{2.56}$$

This statement holds as long as the change $\rho(X)$ is asymptotically smaller than the superpotential $h(X)$ at infinity in the field space.

3 Conclusion

This brief review shows the basic constructions of a QFT in the path integral formalism. The study of supersymmetry on this simplest model clearly reveals some of the fundamental features of supersymmetric theories. In particular, the discussion of localisation principle and deformation invariance of the superpotential is widely applicable to higher dimensional cases. These techniques should form part of the backbone for the future research in higher dimensional supersymmetric quantum field theories.

References

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