NOTE 2020-0314-1835

# Residue Theorem

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Abstract: This is a brief note on the Cauchy's residue theorem and the computation of residues.

*August 26, 2021*

The residue theorem is a powerful tool to evaluate integrals. It states that for a memorophic function $f(z)$ on an open subset $U \subset \mathbb{C}^*$ with isolated singularities at $z = a_i$, the contour integral over a closed counterclockwise directed curve $\gamma$ can be evaluated as [1]

$$\oint_\gamma dz\, f(z) = 2\pi i \sum_i \operatorname{Res}[f(z), a_i]\,, \tag{0.1}$$

where the contour integral on the left hand side is defined by

$$\oint_\gamma dz\, f(z) := \int_{\gamma(t)} d\gamma\, f(\gamma) = \int_{t_0}^{t_1} dt\, \gamma'(t) f[\gamma(t)]\,, \tag{0.2}$$

and the sum on the right hand side is over all singularities enclosed by the curve $\gamma$.

The residue $\operatorname{Res}[f(z), a_i]$ of $f(z)$ at $a_i$ is defined as the order $-1$ coefficient in the Laurent series expansion of $f(z)$ around $a_i$. If we write the Laurent series as

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - a_i)^n\,, \tag{0.3}$$

then the residue is

$$\operatorname{Res}[f(z), a_i] := c_{-1}\,. \tag{0.4}$$

For poles of finite order $k$, there exists a simple closed form formula to compute the residue. By definition, we have the following non-zero and holomorphic function $g(z)$ defined by

$$g(z) := (z - a)^k f(z) \tag{0.5}$$

if and only if the function $f(z)$ has a pole of order $k$ at $z = a$. The holomorphic function $g(z)$ can be Taylor expanded

$$g(z) = \sum_{n=0}^{\infty} \frac{g^{(n)}(a)}{n!} (z - a)^n\,. \tag{0.6}$$

Then the meromorphic function $f(z)$ can be written as

$$f(z) = \sum_{n=0}^{\infty} \frac{g^{(n)}(a)}{n!} (z - a)^{n-k}\,, \tag{0.7}$$

from which the residue can be read off as

$$\operatorname{Res}[f(z), a] = \left. \frac{g^{(k-1)}(z)}{(k-1)!} \right|_{z=a} \tag{0.8}$$

For essential singularities, no such general formulae exist. The residue must be computed from the Laurent series expansion on a case by case basis.

## References


[1] L.V. Ahlfors, *Complex Analysis*, McGraw Hill, New York, USA (1979).
[GoodReader]