NOTE 2020-0326-1543

# Arithmetic Construction of Number Systems

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ABSTRACT: This note briefly describes the construction of number systems, starting from the natural numbers to the complex numbers.

*August 26, 2021*

Most number systems can be built upon extensions to the natural numbers, which are used as both cardinal (counting) and ordinal (ordering) numbers [1]. The natural numbers $\mathbb{N}$ can be defined by the Peano axioms:

1. Zero is a natural number, i.e., $0 \in \mathbb{N}$.
2. The successor $+_1(n)$ of a natural number $n$ is also a natural number, i.e., $+_1(n) \in \mathbb{N}\ \forall n \in \mathbb{N}$.
3. Zero is not the successor to any natural number, i.e., $0 \neq +_1(n)\ \forall n \in \mathbb{N}$.
4. If the successors of two natural numbers are equal, then the two natural numbers are equal, i.e. $+_1(x) = +_1(y) \Rightarrow x = y\ \forall x, y \in \mathbb{N}$.
5. The axiom of induction: if a statement $P(0)$ is true, and $P(n) \Rightarrow P[+_1(n)]$, then $P(n)$ is true for all $n \in \mathbb{N}$.

The arithmetic of natural numbers can be defined by the following rules for any natural numbers $n$ and $k$:

1. $+_k(0) = k$,
2. $+_k[+_1(n)] = +_1[+_k(n)]$,
3. $\times_k(0) = 0$,
4. $\times_k[+_1(n)] = +_k[\times_k(n)]$.

These unary operators $+_k(\bullet)$ and $\times_k(\bullet)$ are symmetric on exchanging $\bullet$ and $k$. They are commonly written as binary operators $+$ and $\times$, i.e., the usual addition and multiplication.

Integers $\mathbb{Z}$ are extended from natural numbers by including the additive inverses of non-zero natural numbers, i.e.,

$$\mathbb{Z} := \mathbb{N} \cup \{m \mid +_n(m) = 0\ \forall n \neq 0 \in \mathbb{N}\}. \tag{0.1}$$

Integers can be further extended by including the multiplicative inverses to generate the set of rational numbers $\mathbb{Q}$:

$$\mathbb{Q} := \mathbb{Z} \cup \{m \mid \times_n(m) = 1\ \forall n \neq 0 \in \mathbb{Z}\}. \tag{0.2}$$

The real number system $\mathbb{R}$ can also be defined by including the limits of all Cauchy sequences, the rational numbers $\mathbb{Q}$ are completed to the real numbers $\mathbb{R}$.

All the above number systems $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ are ordered, which means for any numbers $a$, $b$, $c$, and $d$, we have the follwing properties:

1. Exactly one of $a = b$, $a < b$, and $b < a$ is true.
2. $a < b \Rightarrow a + c < b + c$.
3. $(a < b \text{ and } b < c) \Rightarrow a < c$.
4. $(a < a + b \text{ and } c < d) \Rightarrow b \times c < b \times d$.

This definition of real numbers $\mathbb{R}$ is equivalent to the algebraic axioms, which defines $\mathbb{R}$ as a complete and ordered field.

To further extend the real numbers, we can include the imaginary unit $i$ satisfying $i^2 = -1$ and its all its real multiples. The resulting number system is the complex numbers $\mathbb{C}$, which is a complete but unordered field.

## References


[1] S. Hewson, *A Mathematical Bridge: An Intuitive Journey in Higher Mathematics*, Second Edition, World Scientific, New Jersey, USA (2009). [GoodReader]