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# Chern-Simons Terms from Determinant Line Bundles

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Abstract: This note is on the effective Chern-Simons term generated by integrating out the index bundle. Specifically, the index bundle contributes to the twisted index as the Chern character of its determinant line bundle.

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## 1 Motivation

It is a general phenomenon for 3d theories that when heavy fermions are integrated out, they induce an effective Chern-Simons term as an low-energy effect [1]. It can be understood on the level of lagrangians of vortices in the abelian Higgs model

$$\mathcal{L} = \frac{1}{2} g_{ab(X)} \dot{X}^a \dot{X}^b \,.$$

When considering a bosonic lagrangian without Chern-Simons interactions, adding suitable fermion content and integrating out its “zero modes” produces an effective Chern-Simons term

$$\mathcal{L}_{\text{eff}} = -\kappa \mathcal{A}_a(X) \dot{X}^a \,. \tag{1.1}$$

This has been shown explicitly for a $U(N)$ Yang-Mills Chern-Simons theory coupled to a real adjoint scalar $\sigma$ and scalars $\{q_i \mid i = 1, ..., n_f\}$ in the fundamental representation of the gauge group [1]. The Chern-Simons term in the lagrangian can be reproduced by integrating out chiral multiplets $\{\Phi_i \mid i = 1, ..., N\}$ in the anti-fundamental representation of the $U(N)$ gauge group. The resulting effective Chern-Simons lagrangian from a single chiral multiplet $\Phi$ is given by

$$\mathcal{L}_{\text{eff}} = -\frac{1}{2} \operatorname{sign}(m) \operatorname{Tr}(\omega_a) \dot{X}^a \,, \tag{1.2}$$

in the limit the mass $m \to \infty$. The trace $\operatorname{Tr}(\omega_a)$ is over the the connection $\omega$ on the index bundle, which is the bundle over the vortex moduli space defined by the space of zero mode solutions of the Dirac equation

$$(i\not{D} - (\sigma + m))\Psi = 0 \,,$$

where $\Psi$ is a Dirac fermion in the chiral multiplet $\Phi$.

This formula (1.2) is obtained by integrating out these zero modes. The dynamics of these zero modes is described in terms of the grassmannian coordiantes $\xi^l$ of the fibre of the index bundle by the kinetic term

$$\bar{\xi}^l (iD_t - m) \xi^l \,, \tag{1.3}$$

where the covariant derivative is defined by

$$D_t \xi^l = \partial_t \xi^l + i(\omega_a)^l_n \dot{X}^a \xi^n \,. \tag{1.4}$$

Integrating out the fermion $\xi$ in the path integral leads to the normalised determinant

$$\det\left(\frac{iD_t - m}{i\partial_t - m}\right) = \det\left(\frac{-\partial_\tau - i\omega_a \partial_\tau X^a - m}{-\partial_\tau - m}\right) , \tag{1.5}$$

where $\tau = it$ is the compact euclidean time with periodicity $\tau \in [0, \beta)$.

The solution $\chi$ of eigenvalue $\lambda$ to the equation

$$(-\partial_\tau - i\omega_a \partial_\tau X^a - m)\chi = \lambda \chi \tag{1.6}$$

is given by

$$\chi = e^{-(m+\lambda)\tau} V(\tau) \,, \tag{1.7}$$

where $V(\tau)$ is the time-ordered product

$$V(\tau) = T \exp\left(-i \int_0^\tau d\tau' \, \omega_a \partial_{\tau'} X^a\right) . \tag{1.8}$$

Denoting the eigenvalues of $V(\beta)$ as $e^{\nu_l}$ and imposing the periodicity condition $\chi(0) = \chi(\beta)$ gives

$$\lambda_l = \frac{\nu_l + 2\pi i n}{\beta} - m\,, \quad n \in \mathbb{Z}\,. \tag{1.9}$$

Now the determinant is obtained as

$$\begin{aligned}
&\det\left(\frac{iD_t - m}{i\partial_t - m}\right)\\
=&\prod_l \prod_n \frac{2\pi i n/\beta + \nu_l/\beta - m}{2\pi i n/\beta - m}\\
=&\prod_l \left(1 - \frac{\nu_l}{\beta m}\right) \frac{\sinh(\beta m/2 - \nu_l/2)}{\sinh(\beta m/2)}\\
\to&\exp\left(-\frac{1}{2}\,\mathrm{sign}(m) \sum_l \nu_l\right)
\end{aligned} \tag{1.10}$$

as $\beta \to \infty$. This contribution to the path integral corresponds exactly to the effective lagrangian (1.2). The zero modes have induced an effective magnetic field $\mathcal{F} = d\mathcal{A}$, where $\mathcal{A} = \mathrm{Tr}(\omega)$ is the Chern-Simons one-form. If we integrate out $N = 2\kappa$ chiral multiplets, then the Chern-Simons term (1.1) is recovered.

In general, we would like to identify Chern-Simons terms as effects of some "determinant" of the bundles encoding the chiral multiplets.

## 2 Determinant Line Bundles

### 2.1 Definition

Let $\chi$ be a Deligne-Mumford stack, and $\xi$ be a locally free, finitely generated $O_\chi$ module [2]. The determinant line bundle of $\xi$ is defined as

$$\det(\xi) := \wedge^{\mathrm{rank}(\xi)} \xi\,. \tag{2.1}$$

Let $\mathcal{F}^\bullet$ be a complex of coherent sheaves on $\chi$, which has a bounded complex of locally free, finitely generated $O_\chi$ modules $\mathcal{G}^\bullet$ and a quasi-isomorphism

$$\mathcal{G}^\bullet \to \mathcal{F}^\bullet\,. \tag{2.2}$$

Then the determinant of $\mathcal{F}^\bullet$ is defined as

$$\det(\mathcal{F}^\bullet) := \bigotimes_n \det(\mathcal{G}^n)^{(-1)^n}\,. \tag{2.3}$$

To define the level structure, let $\mathfrak{M}_{g,k}$ be the algebraic stack of pre-stable nodal curves, and $\mathfrak{Bun}_G$ be the relative moduli stack

$$\mathfrak{Bun}_G \xrightarrow{\phi} \mathfrak{M}_{g,k}$$

of principal G-bundles on the fibres of the universal curve $\mathfrak{C}_{g,k} \to \mathfrak{M}_{g,k}$. Given a finite-dimensional representation $R$ of $G$, the level-$l$ determinant line bundle over the $\epsilon$-stable quasi-map space $\mathcal{Q}^\epsilon_{g,k}(Z /\!\!/ G, \beta)$ is defined as

$$\mathcal{D}^{R,l} := (\det(R\pi_*(\mathcal{P} \times_G R)))^{-l}\,, \tag{2.4}$$

where $\mathcal{P} \to \mathcal{C}_{g,k}$ is the universal principal bundle given by the pull-back of the universal principal G-bundle $\tilde{\mathfrak{P}} \to \mathfrak{C}_{\mathfrak{Bun}_{g,k}}$. Here $\mathcal{C}_{g,k} \xrightarrow{\pi} \mathcal{Q}^\epsilon_{g,k}(Z /\!\!/ G, \beta)$ is the universal curve on the quasi-map space.

### 2.2 Abelian Gauge Theory

The twisted index of 3d abelian Chern-Simons matter theories can be interpreted as integrals of characteristic classes over the moduli space of vortices [3]. In particular, the massive fluctuations of a chiral multiplet $\Phi_i$ generate a perfect complex $\mathcal{E}_j^\bullet$ of sheaves defined by the derived push-forward

$$\mathcal{E}_j^\bullet := R^\bullet \pi_*(\mathcal{L}^{Q_j} \otimes \mathcal{K}^{r_j/2}) , \tag{2.5}$$

where $\mathcal{L}$, $\mathcal{K}$ are the universal line bundle and canonical bundle on the product space $\mathfrak{M} \times \Sigma$ of the moduli space $\mathfrak{M}$ and the curve $\Sigma$, and $Q_j$, $r_j$ are the gauge charge and R-charge of $\Phi_j$ respectively. The class $\mathrm{ch}(\mathcal{E}_j^\bullet) = \mathrm{ch}(\mathcal{E}_j^0) - \mathrm{ch}(\mathcal{E}_j^1)$ makes sense in equivariant K-theory and the complex behaves like a vector bundle of rank $d_j - g + 1$ for the purpose of such computations.

Thus the determinant line bundle of $\mathcal{E}_j^\bullet$ can be computed using (2.1) as follows:

$$\det(\mathcal{E}_j^\bullet) = \det(\mathcal{E}_j^0 - \mathcal{E}_j^1) = \wedge^{d_j - g + 1}(\mathcal{E}_j^0 - \mathcal{E}_j^1) . \tag{2.6}$$

The first Chern class of this line bundle is

$$c_1\left(\det(\mathcal{E}_j^\bullet)\right) = c_1\left(\wedge^{d_j - g+1}\mathcal{E}_j^\bullet\right) = c_1\left(\mathcal{E}_j^\bullet\right) . \tag{2.7}$$

#### 2.2.1 Topological Saddles

On the moduli space

$$\mathfrak{M}_m = \mathrm{Pic}^m(\Sigma) \times [\mathrm{pt}/\mathbb{C}^*]$$

of topological saddles, the Chern roots of $\mathcal{E}_j^\bullet$ are

$$(x_1, ..., x_{d_j}) = (0, ..., 0, -Q_j^2\theta_1, ..., -Q_j^2\theta_g) \tag{2.8}$$

where $\theta_a = \zeta_a \wedge \bar{\zeta}_a$ is the wedge product of the standard generators of the cohomology classes on $\Sigma$. The first Chern class is a straightforward evaluation

$$c_1\left(\det(\mathcal{E}_j^\bullet)\right) = \sum_{i=1}^{d_j} x_i = -Q_j^2\theta \tag{2.9}$$

where $\theta = \sum_{a=1}^{g}\theta_a$. For the class of the theories where $Q_i$ is restricted to $\pm 1$, this is simply $-\theta$. If we define the determinant line bundle at level $k$ to be

$$D_k = (\det \mathcal{E}_j^\bullet)^{-k} , \tag{2.10}$$

then its Chern character is given by

$$\mathrm{ch}(D_k) = \exp\left(c_1\left((\det \mathcal{E}_j^\bullet)^{-k}\right)\right) = e^{k\theta} , \tag{2.11}$$

which is exactly the Chern-Simons contribution at level $k$ to the twisted index.

Completing to equivariant forms, the Chern roots of $\mathcal{E}_j^\bullet$ become

$$((Q_j\sigma + Q_F m_j), ..., Q_j\sigma + Q_F m_j, (Q_j\sigma + Q_F m_j - Q_j^2\theta_1), ..., (Q_j\sigma + Q_F m_j - Q_j^2\theta_g)) , \tag{2.12}$$

where $Q_j = \pm 1$ is the gauge charge, and $Q_F$ is the flavour charge.

The first Chern class of $\det\left(\mathcal{E}_j^\bullet\right)$ is then $d_j(Q_j\sigma + Q_F m_j)$. The Chern character of $D_k$ is

$$\begin{aligned} \mathrm{ch}(D_k) &= e^{k\theta} e^{-k d_j (Q_j\sigma + Q_F m_j)} \\ &= e^{k\theta} x^{k d_j Q_j} y_j^{k d_j Q_F} \\ &= e^{k\theta} x^{km} x^{k r_j Q_j (g-1)} y_j^{-k Q_j Q_F m} y_j^{-k r_j Q_F (g-1)} , \end{aligned} \tag{2.13}$$

where $x = e^{-\sigma}$, $y_j = e^{-m_j}$ are the fugacities, and the rank is $d_j = Q_j m + r_j(g-1)$. In the limit of $m_j \to \infty$, this becomes

$$\mathrm{ch}(D_k) = e^{k\theta} x^{km} x^{kr_j Q_j (g-1)} . \tag{2.14}$$

If we identify $kr_j Q_j$ as the mixed Chern-Simons term $k_R$, then this is exactly the Chern-Simons contribution we expected.

### 2.2.2 Vortex Saddles

On the moduli space

$$\mathfrak{M}_m = \sum_{i=1}^{N} \mathrm{Sym}^{d_i} \Sigma$$

of topological saddles, where $i$ labels the single non-vanishing chiral mutiplet $\Phi_i$, the equivariant Chern roots of $\mathcal{E}_j^\bullet$ on each component $\mathrm{Sym}^{d_i}\Sigma$ arising from the chiral multiplet $\Phi_{j\neq i}$ are

$$(Q_j\eta + Q_F m_j, ..., Q_j\eta + Q_F m_j, Q_j\eta + Q_F m_j - Q_j^2\theta_1, ..., Q_j\eta + Q_F m_j - Q_j^2\theta_g) , \tag{2.15}$$

leading to a similar computation

$$\begin{aligned}\mathrm{ch}(D_k) &= e^{k\theta} e^{-kd_j(Q_j\eta + Q_F m_j)} \\ &= e^{k\theta} e^{-kd_j Q_j \eta} y_j^{kd_j Q_F} \\ &= e^{k\theta} e^{-km\eta} e^{-kr_j Q_j (g-1)\eta} y_j^{-kQ_j Q_F m} y_j^{-kr_j Q_F (g-1)} .\end{aligned} \tag{2.16}$$

The generator $\eta$ can be effectively mapped as $e^{-\eta} \to xy_i$, giving the result

$$\mathrm{ch}(D_k) = e^{k\theta} x^{km} x^{kr_j Q_j (g-1)} y_i^{km} y_i^{kr_j Q_j (g-1)} y_j^{-kQ_j Q_F m} y_j^{-kr_j Q_F (g-1)} . \tag{2.17}$$

In the limit $m_j \to \infty$, this again gives

$$\mathrm{ch}(D_k) = e^{k\theta} x^{km} x^{kr_j Q_j (g-1)} , \tag{2.18}$$

which is the same result as (2.14).

Thus we conclude that the Chern-Simons contributions can be obtained by integrating out additional *auxiliary* chiral multiplets.

## References


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