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$\mathrm{U}(1)^K$ Chern-Simons Matter Gauge Theory

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ABSTRACT: This note is about $\mathrm{U}(1)^K$ gauge theory on a topological twist, with Chern-Simons terms and chiral matters.

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1 Introduction

Consider a three-dimensional $\mathcal{N} = 2$ supersymmetric $U(1)^K$ abelian gauge theory with N chiral multiplets Φ_j of gauge charge $Q^a{}_j$ in the a-th U(1) group, and R-charge r_i. For simplicity, the combined gauge and flavour charge matrix is restricted to be unimodular $\det Q^i{}_j = 1$. The superpotential is required to vanish $\mathcal{W} = 0$ to preserve both supersymmetry and global symmetries [1]. We introduce Chern-Simons terms at the following levels:

- gauge Chern-Simons levels κ_{ab},
- mixed gauge-flavour Chern-Simons levels $\kappa_{ab'}$,
- Mixed gauge-R Chern-Simons levels κ_{Ra},
- Mixed flavour-R Chern-Simons levels $\kappa_{Ra'}$,

where the chiral multiplet index is split into $i = (a, a')$, with $a' \in \{K + 1, ..., N\}$ labelling $N - K$ independent flavour fugacities. The gauge and flavour levels can be combined into an extended matrices κ_{ij} and κ_{Ri}. The quantisation condition requires

$$\kappa_{ij} + \frac{1}{2}\sum_{k=1}^{N} Q^i{}_k Q^j{}_k \quad \in \mathbb{Z}\,, \tag{1.1a}$$

$$\kappa_{Ri} + \frac{1}{2}\sum_{k=1}^{N} Q^i{}_k (r_k - 1) \quad \in \mathbb{Z}\,. \tag{1.1b}$$

After integrating out auxiliary fields, the classical scalar potential is obtained as [3]

$$U^{cl} = \sum_{a=1}^{K} e_a^2 \left(\sum_{j=1}^{N} Q^a{}_j |\phi_j|^2 - \sum_{b=1}^{K} \kappa_{ab}\sigma_b - \zeta_a \right)^2 + \sum_{j=1}^{N} M_j^2 |\phi_j|^2 + \sum_{j=1}^{N} \left| \frac{\partial \mathcal{W}}{\partial \phi_j} \right|^2, \tag{1.2}$$

where ϕ_j, σ_a are the scalars in the chiral multiplet scalar and vector multiplet respectively, and ζ_a is the 3d Fayet-Iliopoulis parameter.

The effective mass of ϕ_j is given by

$$M_j = \sum_{a=1}^{K} Q^a{}_j \sigma_a + m_j\,. \tag{1.3}$$

The dynamically generated Chern-Simons terms give corrections to the Chern-Simons levels as

$$\kappa_{ab}^{\text{eff}} = \kappa_{ab} + \frac{1}{2}\sum_{j=1}^{N} Q_{aj} Q_{bj}\, \operatorname{sign} M_j\,, \tag{1.4}$$

and

$$\kappa_{aj}^{\text{eff}} = \frac{1}{2} Q_{aj}\, \operatorname{sign} M_j\,, \tag{1.5}$$

whose combined effects can be interpreted as a renormalisation of the Fayet-Iliopoulis parameter given by

$$\zeta_a^{\text{eff}} = \zeta_a + \sum_{b=1}^{K} \kappa_{ab}^{\text{eff}} \sigma_b + \sum_{j=1}^{N} \kappa_{aj}^{\text{eff}} m_j \tag{1.6}$$

$$= \zeta_a + \sum_{b=1}^{K} \kappa_{ab}\sigma_b + \frac{1}{2}\sum_{j=1}^{N} Q_{aj} |M_j|\,. \tag{1.7}$$

Hence the semi-classical scalar potential is obtained as

$$U = \sum_{a=1}^{K} e_a^2 \left(\sum_{j=1}^{N} Q^a{}_j |\phi_j|^2 - \zeta_a^{\text{eff}} \right)^2 + \sum_{j=1}^{N} M_j^2 |\phi_j|^2 \,. \tag{1.8}$$

The theory can be twisted onto a target space $\Sigma_g \times S^1$, where Σ_g is a Riemann surface of genus g, and S^1 is a circle of radius β. Exponentiating ζ^{eff} gives a factor of the form

$$q_a \prod_{b=1}^{K} x_b^{\kappa_{ab}} \prod_{j=1}^{N} y_j^{\kappa_{aj}} \,,$$

where the fugacities are defined as $q_a := e^{-\beta\zeta_a}$, $x_b := e^{-\beta\sigma_b}$, and $y_j := e^{-\beta m_j}$. The factors of y_j can be absorbed into x_b for $j = 1, ..., K$. This results in factors involving the mixed Chern-Simons terms in the one-loop determinant

$$Z = \left[\prod_{a=1}^{K} q_a^{m_a} \left(\prod_{b=1}^{K} x_b^{\kappa_{ab} m_a} \right) \left(\prod_{b'=K+1}^{N} y_{b'}^{\kappa_{ab'} m_a} \right) \right] \left(\prod_{b=1}^{K} x_b^{\kappa_{Rb}(g-1)} \right) \left(\prod_{b'=K+1}^{N} y_{b'}^{\kappa_{Rb'}(g-1)} \right) \times \\ \left[\prod_{i=1}^{N} \left(\frac{\prod_{b=1}^{K} x_b^{\frac{Q^b{}_i}{2}} \prod_{b'=K+1}^{N} y_{b'}^{\frac{Q^{b'}{}_i}{2}}}{1 - \prod_{b=1}^{K} x_b^{Q^b{}_i} \prod_{b'=K+1}^{N} y_{b'}^{Q^{b'}{}_i}} \right)^{\left(\sum_{c=1}^{K} Q^c{}_i m_c\right) + (r_i - 1)(g-1)} \right] . \tag{1.9}$$

2 Examples

2.1 $U(1)^2$ with Three Chiral Multiplets

Consider a $U(1)^2$ gauge theory with three chiral multiplets. Let the gauge charge matrix $Q^a{}_j$ be

$$Q^a{}_j = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix} , \tag{2.1}$$

and the full charge matrix with flavour charges be

$$Q^i{}_j = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} , \tag{2.2}$$

where the top two rows are the gauge charges, and the third row is the flavour charges.

If we take the mixed gauge-gauge Chern-Simons levels κ_{ab} to be the "critical" levels

$$\kappa_{ab} = \begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix} , \tag{2.3}$$

and real masses to be

$$m_1 = -m \,, \tag{2.4a}$$

$$m_2 = 0 \,, \tag{2.4b}$$

$$m_3 = -m \,, \tag{2.4c}$$

then this theory becomes the $N = 3$ case of the theory A from [3]. The effective masses of ϕ_j are then

$$M_1 = \sigma_1 - m\,, \tag{2.5a}$$
$$M_2 = -\sigma_1 + \sigma_2\,, \tag{2.5b}$$
$$M_3 = -\sigma_2 - m\,. \tag{2.5c}$$

The effective Chern-Simons levels are

$$\kappa^{\text{eff}}_{ab} = \begin{pmatrix} \kappa_{11} + \frac{1}{2}\operatorname{sign} M_1 + \frac{1}{2}\operatorname{sign} M_2 & \kappa_{12} - \frac{1}{2}\operatorname{sign} M_2 \\ \kappa_{21} - \frac{1}{2}\operatorname{sign} M_2 & \kappa_{22} + \frac{1}{2}\operatorname{sign} M_2 + \frac{1}{2}\operatorname{sign} M_3 \end{pmatrix}. \tag{2.6}$$

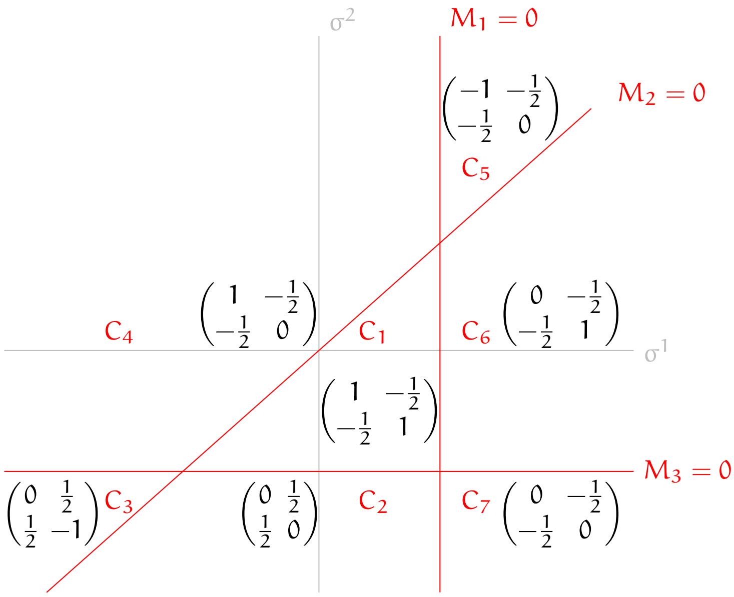
There is a region on the σ_1-σ_2 toric diagram where where κ^{eff}_{ab} vanish.

Given the bare critical Chern-Simons levels in (2.3) and the real masses (2.4), this region is a triangular chamber bounded by

$$\operatorname{sign} M_1 = -1 \quad \Rightarrow \quad \sigma_1 < m\,, \tag{2.7a}$$
$$\operatorname{sign} M_2 = -1 \quad \Rightarrow \quad \sigma_2 < \sigma_1\,, \tag{2.7b}$$
$$\operatorname{sign} M_3 = -1 \quad \Rightarrow \quad \sigma_2 > -m\,. \tag{2.7c}$$

Inside this chamber C_1, the moduli space of the Coulomb branch is a toric variety $\mathbb{CP}^2$. The other six chambers on the toric diagram can be accessed by either shifting the bare Chern-Simons levels. The required critical Chern-Simons levels for each chamber C_1,...,C_7 on the toric diagram are shown in Figure 1.

Figure 1. Critical Chern-Simons Levels on Toric Diagram



The one-loop determinant (1.9) for this theory is then

$$Z = \left(q_1^{m_1} q_2^{m_2}\right)\left(x_1^{\kappa_{11}m_1+\kappa_{21}m_2} x_2^{\kappa_{12}m_1+\kappa_{22}m_2} y_3^{\kappa_{13}m_1+\kappa_{23}m_2}\right)\left(x_1^{\kappa_{R1}(g-1)} x_2^{\kappa_{R2}(g-1)} y_3^{\kappa_{R3}(g-1)}\right) \times \left(\frac{x_1^{\frac{1}{2}}}{1-x_1}\right)^{m_1+(r_1-1)(g-1)} \left(\frac{x_1^{-\frac{1}{2}} x_2^{\frac{1}{2}}}{1-x_1^{-1}x_2}\right)^{-m_1+m_2+(r_2-1)(g-1)} \left(\frac{x_2^{-\frac{1}{2}} y_3^{\frac{1}{2}}}{1-x_2^{-1}y_3}\right)^{-m_2+(r_3-1)(g-1)}. \tag{2.8}$$

2.1.1 Mirror Symmetry

Given some theory A with gauge group $U(1)^K$ and N chiral multiplets, the charges $\tilde{Q}$ in the mirror theory B can be defined in terms of the charges Q of theory A as $\tilde{Q} = (Q^{-1})^t$, satisfying

$$\sum_{i=1}^{N} Q_{ai}\tilde{Q}_{bi} = 0 \tag{2.9}$$

for all $a = 1, ..., K$ and $b = 1, ..., N-K$. This sum vanishes because it is the off-diagonal entries in QQ^{-1}.

With this definition of $\tilde{Q}$, the bottom $N-K$ rows are the gauge charges while the top K rows are flavour charges. To directly compare with Q of theory A, we can swap the rows and columns of the charge matrix $\tilde{Q}$ to put gauge charges on top, while maintaining non-vanishing determinant.

Chamber C_1 For theory A in chamber C_1, the charge matrix Q is given by (2.2). The corresponding $\tilde{Q}$ is

$$(Q^{-1})^t = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

After swapping the orders of the rows and the columns, we have the mirror theory B as a supersymmetric quantum electrodynamics with charges

$$\tilde{Q}^i{}_j = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \tag{2.10}$$

where the first row is the gauge charges. Note that this is not a transposition. With masses all set to $m_i = -\frac{3}{2}$ and Chern-Simons level $k = -\frac{3}{2}$, a Higgs branch opens up at $\sigma = \frac{3}{2}$ with the vortex equation

$$|\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2 - \zeta - k\sigma = 0. \tag{2.11}$$

This gives a moduli space $\mathbb{CP}^2$ provided $\zeta > -k\sigma = \frac{9}{4}$.

Chamber C_2 Similarly, the charges in the mirror for the chamber C_2 are

$$\tilde{Q}^i{}_j = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.12}$$

With Chern-Simons level same mass parameters set to $k = m_i = \frac{1}{2}$, there is a Higgs branch at $\sigma = \frac{1}{2}$, $\phi_3 = 0$, resulting in a vortex equation

$$-|\phi_1|^2 - |\phi_2|^2 - \zeta - k\sigma - \frac{1}{2} = 0. \tag{2.13}$$

The moduli space is the line bundle $\mathbb{CP}^1$ provided $\zeta < -\frac{3}{4}$. The full space with the chiral ϕ_3 of charge -1 can be considered as a line bundle $\mathcal{O}(-1) \to \mathbb{CP}^1$. Alternatively, there is a Higgs branch of $\mathbb{CP}^0 \simeq \{pt\}$ at $\sigma = -\frac{1}{2}$, and $\phi_1 = \phi_2 = 0$, provided $\zeta > \frac{5}{4}$. The full space with the chiral multiplets ϕ_1, ϕ_2 of charge 1 can be considered as $\mathbb{C}^2 \to \mathbb{CP}^0$, which is jsut $\mathbb{C}^2$.

Chamber C_3 The charges in the mirror for the chamber C_3 are

$$\tilde{Q}^i{}_j = \begin{pmatrix} 1 & -1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.14}$$

Similar to the case in C_2, with Chern-Simons level same mass parameters set to $k = m_i = -\frac{1}{2}$, there is a Higgs branch at $\sigma = -\frac{1}{2}$, $\phi_1 = \phi_3 = 0$, resulting in a vortex equation

$$-|\phi_2|^2 - \zeta - k\sigma - 1 = 0. \tag{2.15}$$

The moduli space is $\mathbb{C}^2$ provided $\zeta < -\frac{5}{4}$. Alternatively, there is a Higgs branch of $\mathcal{O}(-1) \to \mathbb{CP}^1$ at $\sigma = \frac{1}{2}$ provided $\zeta > \frac{3}{4}$.

2.1.2 Bethe Ansats Equations

Taking the one-loop determinant (2.8) for the theory A in C_1, the exponent $iB_a = \frac{\partial \log Z}{\partial m_a}$ in the Bethe ansatz equations [2] are

$$iB_1 = \log q_1 + \kappa_{11} \log x_1 + \kappa_{12} \log x_2 + \kappa_{13} \log y_3 + \log\left(\frac{x_1^{\frac{1}{2}}}{1 - x_1}\right) - \log\left(\frac{x_1^{-\frac{1}{2}} x_2^{\frac{1}{2}}}{1 - x_1^{-1} x_2}\right), \tag{2.16a}$$

$$iB_2 = \log q_2 + \kappa_{21} \log x_1 + \kappa_{22} \log x_2 + \kappa_{23} \log y_3 + \log\left(\frac{x_1^{-\frac{1}{2}} x_2^{\frac{1}{2}}}{1 - x_1^{-1} x_2}\right) - \log\left(\frac{x_2^{-\frac{1}{2}} y_3^{\frac{1}{2}}}{1 - x_2^{-1} y_3}\right). \tag{2.16b}$$

The Bethe ansatz equations $e^{iB_a} = 1$ are

$$1 = e^{iB_1} = q_1 x_1^{\kappa_{11}} x_2^{\kappa_{12}} y_3^{\kappa_{13}} \left(\frac{x_1^{\frac{1}{2}}}{1 - x_1}\right) \left(\frac{x_1^{-\frac{1}{2}} x_2^{\frac{1}{2}}}{1 - x_1^{-1} x_2}\right)^{-1}, \tag{2.17a}$$

$$1 = e^{iB_2} = q_2 x_1^{\kappa_{12}} x_2^{\kappa_{22}} y_3^{\kappa_{23}} \left(\frac{x_1^{-\frac{1}{2}} x_2^{\frac{1}{2}}}{1 - x_1^{-1} x_2}\right) \left(\frac{x_2^{-\frac{1}{2}} y_3^{\frac{1}{2}}}{1 - x_2^{-1} y_3}\right)^{-1}. \tag{2.17b}$$

Rearranging and relabelling $y_3 \mapsto y$ gives

$$q_1 = x_1^{-\kappa_{11}-1} x_2^{-\kappa_{12}+\frac{1}{2}} (1 - x_1)(1 - x_1^{-1} x_2)^{-1} y^{-\kappa_{13}}, \tag{2.18a}$$

$$q_2 = x_1^{-\kappa_{21}+\frac{1}{2}} x_2^{-\kappa_{22}-1} (1 - x_1^{-1} x_2)(1 - x_2^{-1} y)^{-1} y^{-\kappa_{23}+\frac{1}{2}}. \tag{2.18b}$$

We can send $x_1 \mapsto x_1^{-1}$, $x_2 \mapsto x_2^{-1}$, and $y \mapsto y^{-1}$ to examine solutions at infinity. The effect is flipping the signs on Chern-Simons levels, up to an overall factor of $(-1)^K$. The results for this case $K = 2$ are

$$q_1 = x_1^{\kappa_{11}-1} x_2^{\kappa_{21}+\frac{1}{2}} (1 - x_1)(1 - x_1^{-1} x_2)^{-1} y^{\kappa_{13}}, \tag{2.19a}$$

$$q_2 = x_1^{\kappa_{12}+\frac{1}{2}} x_2^{\kappa_{22}-1} (1 - x_1^{-1} x_2)(1 - x_2^{-1} y)^{-1} y^{\kappa_{23}+\frac{1}{2}}, \tag{2.19b}$$

which are equivalent to (2.18). Substituting in the "critical" Chern-Simons levels from (2.3) produces

$$q_1 = (1 - x_1)(1 - x_1^{-1} x_2)^{-1} y^{\kappa_{13}}, \tag{2.20a}$$

$$q_2 = (1 - x_1^{-1} x_2)(1 - x_2^{-1} y)^{-1} y^{\kappa_{23}+\frac{1}{2}}. \tag{2.20b}$$

To eliminate factors of y, we can set the mixed gauge-flavour Chern-Simons levels to be critical at $k_{13} = 0$ and $k_{23} = -\frac{1}{2}$, i.e.,

$$\kappa_{ij} = \begin{pmatrix} 1 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 1. \end{pmatrix}. \tag{2.21}$$

Then the Bethe ansats equations become

$$q_1 = (1 - x_1)(1 - x_1^{-1}x_2)^{-1}, \tag{2.22a}$$
$$q_2 = (1 - x_1^{-1}x_2)(1 - x_2^{-1}y)^{-1}. \tag{2.22b}$$

2.1.3 Twisted Index

For the theory A in C_1, the hessian factors

$$H_a = \frac{\partial^2 \log Z}{\partial \log x_a \partial m_a} = \frac{\partial i B_a}{\partial \log x_a} \tag{2.23}$$

are computed to be

$$H_1 = \kappa_{11} + 1 + \frac{x_1}{1 - x_1} + \frac{x_1^{-1}x_2}{1 - x_1^{-1}x_2}, \tag{2.24a}$$
$$H_2 = \kappa_{22} + 1 + \frac{x_1^{-1}x_2}{1 - x_1^{-1}x_2} + \frac{x_2^{-1}y}{1 - x_2^{-1}y}. \tag{2.24b}$$

The contour integral formula for the twisted index is then

$$\mathcal{I} = \sum_{m_1} \sum_{m_2} \left(\frac{1}{2\pi i}\right)^2 \oint_{JK} \frac{dx_1}{x_1}\frac{dx_2}{x_2} H_1^g H_2^g Z(x_1, x_2; m_1, m_2), \tag{2.25}$$

where the integral is evaluated with the Jeffrey-Kirwan prescription.

Consider the case $g = 0$ where the hessian factor can be ignored. Take the gauge Chern-Simons levels to be the "critical" levels in (2.3). Set R-charges to $r_i = 0$ to compare with the mirror theory of R-charges $\tilde{r}_i = 1$. The one-loop determinant (2.8) becomes

$$\begin{aligned} Z =& \left(q_1^{m_1} q_2^{m_2}\right)\left(x_1^{m_1 - \frac{1}{2}m_2} x_2^{-\frac{1}{2}m_1 + m_2} y_3^{\kappa_{13}m_1 + \kappa_{23}m_2}\right)\left(x_1^{-\kappa_{R1}} x_2^{-\kappa_{R2}} y_3^{-\kappa_{R3}}\right) \times \\ & \left(\frac{x_1^{\frac{1}{2}}}{1 - x_1}\right)^{m_1+1} \left(\frac{x_1^{-\frac{1}{2}} x_2^{\frac{1}{2}}}{1 - x_1^{-1}x_2}\right)^{-m_1+m_2+1} \left(\frac{x_2^{-\frac{1}{2}} y_3^{\frac{1}{2}}}{1 - x_2^{-1}y_3}\right)^{-m_2+1}. \end{aligned} \tag{2.26}$$

The contour integral is then

$$\begin{aligned} \mathcal{I} = \sum_{m_1, m_2} & q_1^{m_1} q_2^{m_2} y_3^{\kappa_{13}m_1 + (\kappa_{23} - \frac{1}{2})m_2 + \frac{1}{2} - \kappa_{R3}} \left(\frac{1}{2\pi i}\right)^2 \oint_{JK} \frac{dx_1}{x_1}\frac{dx_2}{x_2} \\ & x_1^{2m_1 - m_2 - \kappa_{R1}} x_2^{-m_1 + 2m_2 - \kappa_{R2}} \left(\frac{1}{1 - x_1}\right)^{m_1+1} \left(\frac{1}{1 - \frac{x_2}{x_1}}\right)^{-m_1+m_2+1} \left(\frac{1}{1 - \frac{y_3}{x_2}}\right)^{-m_2+1}. \end{aligned} \tag{2.27}$$

The Jeffrey-Kirwan charges (Q_1, Q_2) for the three denominator factors are respectively

$$\begin{aligned} Q^{x_1=1} &= (1, 0), \\ Q^{x_1=x_2} &= (-1, 1), \\ Q^{x_2=y_3} &= (0, -1), \end{aligned}$$

which are responsible for the the interior poles at $(1,1)$, $(1,y_3)$, and (y_3,y_3). Taking $\eta=(1,1)$ selects only the residue at $(x_1,x_2)=(1,1)$.

However, for $\mathcal{N}=2$ theory there are potentially additional residues from the topological vacua at $(0,0)$, $(0,y_3)$, $(0,\infty)$, $(1,0)$, $(1,y_3)$, $(1,\infty)$, $(\infty,0)$, (∞,y_3), and (∞,∞). The Jeffrey-Kirwan charges for the x_a factor can be assigned analogously to the $U(1)$ theory [6] as follows

$$Q_b^{x_a=0}=-\delta_{ab}\kappa_{aa}^{\text{eff}}(\sigma_a\mapsto\infty)\,, \tag{2.28a}$$
$$Q_b^{x_a=\infty}=\delta_{ab}\kappa_{aa}^{\text{eff}}(\sigma_a\mapsto-\infty)\,. \tag{2.28b}$$

The resulting normalised charge vectors for this $U(1)^2$ theory are

$$\begin{aligned}Q^{x_1=0}&=(-1,0)\,,\\ Q^{x_1=\infty}&=(1,0)\,,\\ Q^{x_2=0}&=(0,-1)\,,\\ Q^{x_2=\infty}&=(0,1)\,.\end{aligned}$$

Taking into account both interior and boundary poles, $\eta=(1,1)$ selects residues at $(1,1)$, $(1,\infty)$, and (∞,∞).

The residues can be computed following the procedure in [4] with the Mathematica package from [5]. The Jeffrey-Kirwan charge vectors are ordered anti-clockwise. For simplicity, these mixed Chern-Simons levels κ_{R1} and κ_{R2} are set to zero, which only shifts the residues at boundaries into different magnetic sectors. The index is then

$$\frac{y_3^{\frac{1}{2}-\kappa_{R3}}}{1-y_3}\,, \tag{2.29}$$

which is simply the zeroth sector contribution from the pole at $(1,1)$.

In the mirror theory B, the classical and one-loop determinant at $r_i=1$ is

$$Z=\tilde{q}^{\tilde{m}}\tilde{x}^{\tilde{\kappa}_{11}\tilde{m}}\tilde{y}_2^{\tilde{\kappa}_{12}\tilde{m}}\tilde{y}_3^{\tilde{\kappa}_{13}\tilde{m}}\left(\tilde{x}_1^{-\tilde{\kappa}_{R1}}\tilde{y}_2^{-\tilde{\kappa}_{R2}}\tilde{y}_3^{-\tilde{\kappa}_{R3}}\right)\left(\frac{\tilde{x}^{\frac{1}{2}}}{1-\tilde{x}}\right)^{\tilde{m}}\left(\frac{\tilde{x}^{\frac{1}{2}}\tilde{y}_2^{\frac{1}{2}}}{1-\tilde{x}\tilde{y}_2}\right)^{\tilde{m}}\left(\frac{\tilde{x}^{\frac{1}{2}}\tilde{y}_2^{\frac{1}{2}}\tilde{y}_3^{\frac{1}{2}}}{1-\tilde{x}\tilde{y}_2\tilde{y}_3}\right)^{\tilde{m}}. \tag{2.30}$$

To compare with the mirror, set the gauge-flavour Chern-Simons levels to be critical at $\tilde{\kappa}_{11}=-\frac{3}{2}$, $\tilde{\kappa}_{12}=-1$, and $\tilde{\kappa}_{13}=-\frac{1}{2}$. For simplicity, set $\tilde{\kappa}_{R1}=0$. The index at $g=0$, $r_i=1$ is then

$$-\frac{\tilde{q}\tilde{y}_2^{-\kappa_{R2}}\tilde{y}_3^{-\kappa_{R3}}}{1-\tilde{q}}\,. \tag{2.31}$$

With the mirror map $\tilde{q}\to 1/y_3$, it becomes

$$\frac{\tilde{y}_2^{-\kappa_{R2}}\tilde{y}_3^{-\kappa_{R3}}}{1-y_3}\,, \tag{2.32}$$

which matches (2.29) by setting

$$\begin{aligned}\kappa_{R3}&=\frac{1}{2}\,,\\ \tilde{\kappa}_{R2}&=0\,,\\ \tilde{\kappa}_{R3}&=0\,.\end{aligned}$$

2.2 $U(1)$ with Two Chiral Multiplets

Now consider simpler $U(1)$ mirror theories with two chiral multiplets. Take theory A with charge matrix

$$Q^i{}_j = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \tag{2.33}$$

masses $m_1 = m_2 = -m \leqslant 0$, and gauge Chern-Simons level $\kappa_{11} = 1$ critical at $-m \leqslant \sigma \leqslant m$. The Coulomb branch opens at $\sigma \leqslant -m$ for $\zeta = 0$. The mirror theory B has charge matrix

$$\tilde{Q}^i{}_j = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \tag{2.34}$$

masses $\tilde{m}_1 = \tilde{m}_2 = -\tilde{m} \leqslant 0$, and gauge Chern-Simons level $\tilde{\kappa}_{11} = -1$ critical at $\tilde{\sigma} \geqslant \tilde{m}$. The Higgs branch opens at $\tilde{\sigma} = \tilde{m}$.

The one-loop determinants are respectively

$$Z^A = q^m x^m y_2^{\kappa_{12} m} x^{\kappa_{R1}(g-1)} y_2^{\kappa_{R2}(g-1)} \left(\frac{x^{\frac{1}{2}}}{1-x} \right)^{m+(r_1-1)(g-1)} \left(\frac{x^{-\frac{1}{2}} y_2^{\frac{1}{2}}}{1-x^{-1} y_2} \right)^{m+(r_2-1)(g-1)} , \tag{2.35}$$

and

$$Z^B = \tilde{q}^{\tilde{m}} \tilde{x}^{-\tilde{m}} \tilde{y}_2^{\tilde{\kappa}_{12} \tilde{m}} \tilde{x}^{\tilde{\kappa}_{R1}(g-1)} \tilde{y}_2^{\tilde{\kappa}_{R2}(g-1)} \left(\frac{\tilde{x}^{\frac{1}{2}}}{1-\tilde{x}} \right)^{\tilde{m}+(\tilde{r}_1-1)(g-1)} \left(\frac{\tilde{x}^{\frac{1}{2}} \tilde{y}_2^{\frac{1}{2}}}{1-\tilde{x}\tilde{y}_2} \right)^{\tilde{m}+(\tilde{r}_2-1)(g-1)} . \tag{2.36}$$

For simplicity, $\kappa_{R1} = \tilde{\kappa}_{R1} = 0$ is taken, which only shifts the boundary residues into different magnetic sectors. Taking $\eta = 1$ for both theories selects the poles at $x = 1$, $x = \infty$, $\tilde{x} = 1$, and $\tilde{x} = \tilde{y}_2^{-1}$. The twisted indices at $g = 0$ are then computed to be

$$\mathcal{I}^A_{r_i=0} = -\frac{y^{\frac{1}{2}+\kappa_{R2}}}{1-y}, \tag{2.37}$$

and

$$\mathcal{I}^B_{r_i=1} = -\frac{\tilde{q}\tilde{y}^{\frac{1}{2}+\tilde{\kappa}_{12}+\tilde{\kappa}_{R2}}}{1-\tilde{q}\tilde{y}^{\frac{1}{2}\tilde{\kappa}_{12}}} \tag{2.38}$$

$$\mapsto -\frac{y q^{\frac{1}{2}+\tilde{\kappa}_{12}+\tilde{\kappa}_{R2}}}{1-y q^{\frac{1}{2}\tilde{\kappa}_{12}}}, \tag{2.39}$$

where $\mathcal{I}^B_{r_i=1}$ is mapped under $\tilde{q} \mapsto y$ and $\tilde{y} \mapsto q$ in the last line. They agree with each other when $\kappa_{R2} = \frac{1}{2}$, $\tilde{\kappa}_{12} = -\frac{1}{2}$, and $\tilde{\kappa}_{R2} = 0$.

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