NOTE 2021-0813-1008

Multivariate Jeffrey-Kirwan Residue

Guangyu Xu

E-mail: guangyu.xu@cantab.ac.uk

ABSTRACT: This note is on the relation between the multivariate residue and the Jeffrey-Kirwan residue prescription. The Jeffrey-Kirwan prescription is conjectured to determine the partitioning of denominator factors in the integrand of the multivariate residue. The conjecture is applied to the computation of the twisted indices of a mirror pair of $\mathcal{N} = 4$ gauge theories in three dimensions.

August 25, 2021

1 Multivariate Residue

Consider a meromorphic n-form

$$\omega = \frac{h(z)\, dz_1 \wedge ... \wedge dz_n}{f_1(z)...f_n(z)}\,, \tag{1.1}$$

where $h(z) : \mathbb{C}^n \to \mathbb{C}$ and $f(z) = (f_1(z), ..., f_n(z)) : \mathbb{C}^n \to \mathbb{C}^n$ are holomophic functions.

Definition 1.1. A pole of the meromorphic n-form ω is a point $p \in \mathbb{C}^n$ where f has an isolated zero, i.e.,

$$f(p) = 0\,, \tag{1.2}$$

and

$$f^{-1}(0) \cap U = p\,, \tag{1.3}$$

for a sufficiently small neigbourhood U of p [1].

Definition 1.2. The residue at a pole p is defined as an integral over a product of n circles, i.e., an n-torus,

$$\text{Res}_p(\omega) := \frac{1}{(2\pi i)^n} \oint_{\Gamma_\epsilon} \frac{h(z)\, dz_1 \wedge ... \wedge dz_n}{f_1(z)...f_n(z)}\,, \tag{1.4}$$

where $\Gamma_\epsilon := \{z \in \mathbb{C}^n \mid |f_i(z)| = \epsilon_i\}$ is the pre-image of an n-torus under f, and the integration cycle is oriented such that

$$d(\arg f_1) \wedge ... \wedge d(\arg f_n) \geqslant 0. \tag{1.5}$$

It can be generalised to the case where there are different number of denominator factors $f(z) = (f_1(z), ..., f_m(z))$ than the number n of variables. For $m < n$, the relevant construction is called a residual form. For $m > n$, the denominator factors need to be grouped into exactly n partitions. The residue also depends the partitioning, in addition to the pole. Hence, unlike univariate residues, a generic multivariate residue is not solely determined by the pole.

When the jacobian determinant at a pole p

$$J(p) := \det_{i,j}\left(\frac{\partial f_i}{\partial z_j}\right)\Bigg|_{z=p} \tag{1.6}$$

is non-vanishing, the residue is said to be non-degenerate. Non-degenerate residues can be directly evaluated by a coordinate transformation $u = f(z)$ as [2]

$$\text{Res}_p(\omega) = \frac{1}{(2\pi i)^n} \oint_{|u_i| \leqslant \epsilon_i} \frac{h(f^{-1}(u))\, du_1 \wedge ... \wedge du_n}{J(p) u_1(z)...u_n(z)} = \frac{h(p)}{J(p)}\,. \tag{1.7}$$

However, this formula immediately breaks down for higher order poles as they are degenerate.

To evaluate a generic residue, we need to utilise the transformation law [3, p.657-658], which is general property of residues.

Theorem 1.1 (Transformation Law)**.** Let $I = \langle f_1(z), ..., f_n(z)\rangle$ be the ideal generated by a finite set of holomorphic functions f_i such that the solution to

$$f_1(z) = ... = f_n(z) = 0$$

is a finite set of points $\{p, ..., q\}$, i.e., zero-dimensional. Suppose the zero-dimensional ideal $J = \langle g_1(z), ..., g_n(z)\rangle$ is a subspace $J \subseteq I$. Then J is related to I by a holomorphic matrix A such that

$$g_i(z) = \sum_j a_{ij} f_j(z)\,. \tag{1.8}$$

Then the residue at p satisfies

$$\text{Res}_p\left(\frac{h(z)\,dz_1\wedge\ldots\wedge dz_n}{f_1(z)\ldots f_n(z)}\right)=\text{Res}_p\left(\frac{\det(A)(z)h(z)\,dz_1\wedge\ldots\wedge dz_n}{g_1(z)\ldots g_n(z)}\right). \tag{1.9}$$

The transformation formula (1.9) can simplify the computation of a multivariate residue to the product of univariate residues, by choosing all of $g_1,\ldots,g_n$ to be univariate. A set of these univariate polynomials can be obtained from the Gröbner bases of $\{f_1(z),\ldots,f_n(z)\}$ with different lexicographic monomial orders.

A univariate polynomial $g_i(z_i)$ in z_i is taken to be the first element of the Gröbner basis generated with the order $z_{i+1}\succ z_{i+2}\succ\ldots\succ z_n\succ z_1\succ\ldots\succ z_i$. By computing all cyclic permutations of this ordering, we obtain a set of n univariate polynomials $\{g_1(z_1),\ldots,g_n(z_n)\}$.

The transformation matrix A can be obtained using the algorithm implemented in [4]. Shown below is the Mathematica code for two ideal generators in two variables. The matrix A is assembled row by row by taking the first row of `tT` with the corresponding lexicographic ordering. An improved version of this method is used by the function `MultivariateResidue` [1].

```
moduleGroebnerBasis[polys_, vars_, cvars_, opts___] := Module[
  {newpols, rels, len = Length[cvars], gb, j, k, ruls},
  rels = Flatten[Table[cvars[[j]]*cvars[[k]], {j, len}, {k, j, len}]];
  newpols = Join[polys, rels];
  gb = GroebnerBasis[newpols, Join[cvars, vars], opts];
  rul = Map[(# :> {}) &, rels];
  gb = Flatten[gb /. rul];
  Collect[gb, cvars]
  ]

fF = {f[1], f[2]} (* set of ideal generators *)
vars = {x_2, x_1}; (* lexicographic ordering of variables *)

(* encode positions of ideal generators in a matrix *)
coords = Array[ee, 3];
fmat = {{fF[[1]], 1, 0}, {fF[[2]], 0, 1}};
newfF = fmat . coords;

mgb = moduleGroebnerBasis[newfF, vars, coords];
mgb = Select[mgb, Coefficient[#1, coords[[1]]] =!= 0 &];
gG = (Coefficient[#1, coords[[1]]] &) /@ mgb (* groebner basis *)
sS = First /@ PolynomialReduce[fF, gG, vars];

(* check with built-in groebner basis *)
gb = GroebnerBasis[fF, vars];
gb === gG
rul = {ee[1] -> 1, ee[2] -> -fF[[1]], ee[3] -> -fF[[2]]};
Map[Expand[# /. rul] &, mgb] (* want zeroes *)
Expand[sS . gG - fF] (* want zeroes *)

tT = Outer[D, mgb, Rest[coords]]
Expand[gG - tT . fF] (* want zeroes *)
```

2 Jeffrey-Kirwan Prescription

Consider the n-form

$$\omega = \frac{h(z)\, dz_1 \wedge ... \wedge dz_n}{f_1(z)...f_m(z)}\,, \tag{2.1}$$

where $m \geqslant n$, and f_i are linear functions in $z_1, ..., z_n$. Following [5], the Jeffrey-Kirwan residue of ω is defined as

$$\text{JK-Res}_p(\omega) = \frac{1}{|J(p)_{f_i \in F}|} \frac{h(p)}{\prod_{f_i \notin F} f_i}\,, \tag{2.2}$$

where F is a set of exactly n factors responsible for the pole at p. Given a fixed charge vector η, the Jeffrey-Kirwan prescription dictates to only take the sum of residues at those poles such that the charge vectors of the responsible factors form a convex cone containing η. Moreover, the final result is independent of the choice of η.

However, this formula is only valid if all denominator factors are linear functions. Hence it is desirable to build a dictionary to evaluate Jeffrey-Kirwan residues in terms of multivariate residues. The idea is that the Jeffrey-Kirwan prescription determines the partitioning of denominator factors.

Conjecture 2.1. The following procedure is conjectured to compute the Jeffrey-Kirwan residue at a given contributing pole p.

1. Determine the denominator factors $f_1, ..., f_n$ responsible for this pole.
2. Split the denominator factors into n partitions such that each responsible factor is placed in a separate partition.
3. Compute the multivariate residue of the resulting n-form using the transformation law (1.9).

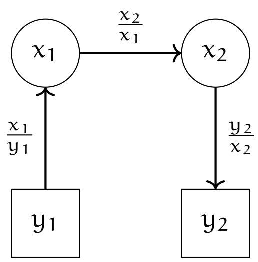
The result of the computation gives the individual Jeffrey-Kirwan residue up to a sign, which is determined by the ordering of the denominator factors.

For two-forms, we can order the charge vectors of the denominator factors by their polar angles. This gives the correct combination of residues as in the Jeffrey-Kirwan prescription. However, for higher forms, it is yet to know how to order the charge vectors correctly.

3 Example

Consider the abelian A_2 linear quiver gauge theory with $\mathcal{N} = 4$ in three dimensions, as shown in Fig 3. At genus $g = 0$, the B-twisted index $Z^{[A_2]}_{g=0,B}$ is expected to be identical to the A-twisted index $Z^{\text{SQED}[3]}_{g=0,A}$ of the quantum electrodynamics of three hypermultiplets in the mirror, and vice versa.

Figure 1. A_2 Linear Quiver Diagram



3.1 B-Twist of Abelian Linear Quiver

First consider the mirror of the A_2 linear quiver, which is the supersymmetric quantum electrodynamics with three hypermultiplets. The contour integral formula for the A-twisted index is [6, (6.35)]

$$Z^{\mathrm{SQED}[3]}_{g=0,A} = -(t-t^{-1})^{-1}\sum_m q^m \frac{1}{2\pi i}\oint_{\mathrm{JK}} \frac{dx}{x}\left(\frac{x-y_1t}{y_1-xt}\right)^m \left(\frac{x-y_2t}{y_2-xt}\right)^m \left(\frac{x-y_3t}{y_3-xt}\right)^m . \tag{3.1}$$

Taking $\eta = 1$ selects the residues at y_1/t, y_2/t, and y_3/t, which sum to [6, (6.40)]

$$Z^{\mathrm{SQED}[3]}_{g=0,A} = \frac{t^{-1}(1-t^{-6})}{(1-t^{-2})(1-qt^{-3})(1-q^{-1}t^{-3})} . \tag{3.2}$$

After the change of variables $t \mapsto t^{-1}$, and $q \mapsto \frac{y_1}{y_2}$, the A-twisted index reads

$$Z^{\mathrm{SQED}[3]}_{g=0,A} = -\frac{(t+t^3+t^5)y_1y_2}{(t^3y_1-y_2)(t^3y_2-y_1)} . \tag{3.3}$$

We would like to compute the B-twisted index $Z^{[A_2]}_{g=0,B}$ of the quiver from Jeffrey-Kirwan prescription and verify if it is identical to this expression.

The contour integral formula of the index $Z^{[A_2]}_{g=0,B}$ is

$$\begin{aligned} Z^{[A_2]}_{g=0,B} =& (t-t^{-1})^2 \sum_{m_1,m_2} q_1^{m_1} q_2^{m_2} \frac{1}{(2\pi i)^2}\oint_{\mathrm{JK}} \frac{dx_1}{x_1}\frac{dx_2}{x_2} \\ & \left(\frac{x_1-y_1t}{y_1-x_1t}\right)^{m_1}\left(\frac{x_2-y_2t}{y_2-x_2t}\right)^{m_2}\left(\frac{x_1-x_2t}{x_2-x_1t}\right)^{m_1-m_2} \times \\ & \frac{x_1y_1t}{(x_1-y_1t)(y_1-x_1t)}\frac{x_2y_2t}{(x_2-y_2t)(y_2-x_2t)}\frac{x_1x_2t}{(x_2-x_1t)(x_1-x_2t)} . \end{aligned} \tag{3.4}$$

The only contributing sector is expected to be the zeroth sector $m_1 = 0, m_2 = 0$, which has the integrand

$$\omega_{0,0} = \frac{t(1-t^2)^2 y_1y_2x_1x_2\, dx_1 dx_2}{(x_1-y_1t)(y_1-x_1t)(x_2-y_2t)(y_2-x_2t)(x_2-x_1t)(x_1-x_2t)} . \tag{3.5}$$

The six denominator factors have charge vectors (Q_1, Q_2) listed below

$$(-1,0),(1,0),(0,-1),(0,1),(1,-1),(-1,1) . \tag{3.6}$$

Note that its Jeffrey-Kirwan residue can still be computed using (2.2).

Given the charge vector $\eta = (1,1)$, the Jeffrey-Kirwan prescription picks the following three poles

$$(y_1/t, y_2/t), (y_1t, y_1/t^2), (y_2/t^2, y_2/t) ,$$

whose charge vectors form cones containing η.

Consider first the pole at $(y_1/t, y_2/t)$ for the procedure in Conjecture 2.1. The responsible denominator factors are $(y_1 - x_1t), (y_2 - x_2t)$. We then split all denominator factors into two partitions $\{f_1, f_2\}$, each containing one responsible factor, say

$$\{f_1, f_2\} = \{(y_1-x_1t), (y_2-x_2t)(x_1-y_1t)(x_2-y_2t)(x_2-x_1t)(x_1-x_2t)\} .$$

Following the transformation law 1.1, first we compute the Gröbner bases of f_1, f_2. Taking the order $x_2 \succ x_1$ gives a basis whose first entry only depends on x_1, while taking the order $x_1 \succ x_2$

gives another basis whose first entry depends only on x_2. These two entries are then taken to be the new denominator factors g_1, g_2

$$\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \begin{bmatrix} -y_1 + x_1 t \\ (-1+t^2)y_1(-y_1^2 + (1+t^2)y_1x_2 - t^2x_2^2)(y_2x_2 + t^2y_2x_2 - t(y_2^2 + x_2^2)) \end{bmatrix} \tag{3.7}$$

The transformation matrix A taking f_1, f_2 to g_1, g_2 is found to be

$$A = \begin{pmatrix} -1 & 0 \\ a_{21} & -t^2 \end{pmatrix}, \tag{3.8}$$

where $a_{21} = (y_2x_2 + t^2y_2x_2 - t(y_2^2 + x_2^2))(-((-1+t^2)y_1^2) + (-1+t^2)y_1(-tx_1 + x_2 + t^2x_2) + t(-x_1x_2 - t^2x_1x_2 + t(x_1^2 + x_2^2)))$, satisfying

$$A \cdot \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}.$$

The determinant $\det(A) = t^2$, giving a transformed integrand

$$\begin{aligned} \omega'_{0,0} &= \frac{\det(A)t(1-t^2)^2 y_1y_2x_1x_2\, dx_1 dx_2\, dx_1 dx_2}{g_1 g_2} \\ &= \frac{t^3(-1+t^2)y_2x_1x_2\, dx_1 dx_2}{(y_1 - tx_1)(y_1 - x_2)(y_1 - t^2x_2)(y_2x_2 + t^2y_2x_2 - t(y_2^2 + x_2^2))}. \end{aligned} \tag{3.9}$$

Now its residue can be evaluated as the product of two univariate residues.

Hence, the multivariate residue at $(y_1/t, y_2/t)$ with respect to this partitioning $\{f_1, f_2\}$ is computed to be

$$\mathrm{Res}_{(y_1/t, y_2/t)}(\omega_{0,0}) = \frac{t y_1 y_2}{(ty_1 - y_2)(ty_2 - y_1)}. \tag{3.10}$$

Note that the result flips signs if we exchange f_1 and f_2 in the partitioning. To get the correct combination of Jeffrey-Kirwan residues, we order the denominator factors such that their corresponding charge vectors are in anti-clockwise order.

Similarly, the residues at $(y_1 t, y_1/t^2)$ and $(y_2/t^2, y_2/t)$ are

$$\mathrm{Res}_{(y_1 t, y_1/t^2)}(\omega_{0,0}) = -\frac{t^2 y_1 y_2}{(y_1 - ty_2)(y_1 - t^3 y_2)} \tag{3.11}$$

and

$$\mathrm{Res}_{(y_2/t^2, y_2/t)}(\omega_{0,0}) = -\frac{t^2 y_1 y_2}{(ty_1 - y_2)(t^3 y_1 - y_2)}. \tag{3.12}$$

The sum of the three residues is

$$Z^{\mathrm{SQED}[3]}_{g=0,A} = \frac{(t + t^3 + t^5) y_1 y_2}{(t^3 y_1 - y_2)(t^3 y_2 - y_1)}, \tag{3.13}$$

which reproduces the expected result in (3.3) up to an overall minus sign. This minus sign could also have been obtained if we picked clockwise ordering for the charge vectors instead of the anti-clockwise ordering.

It can also be verified that the sectors of $m_1 \neq 0$ or $m_2 \neq 0$ have residues summing to zero from these three poles.

Although the computation is done for the particular choice of $\eta = (1, 1)$, it can be shown that the result does not depend on this choice, as long as η is not chosen to align with any of the charges (Q_1, Q_2) in (3.6). This can be explicitly seen by plotting the charges on the Q_1-Q_2 plane, and observe that any choice of η is contained within three convex cones, each giving one of the factors in (3.10), (3.11), and (3.12).

3.2 A-Twist of Abelian Linear Quiver

For the mirror theory, the contour integral formula [6, (6.44)] for the B-twisted index of quantum electrodynamics with three hypermultiplets is

$$Z^{\mathrm{SQED}[3]}_{g=0,B} = -(t-t^{-1})\sum_m (-q)^m \frac{1}{2\pi i}\oint_{\mathrm{JK}} \frac{dx}{x} \left(\frac{x-y_1t}{y_1-xt}\right)^m \left(\frac{x-y_2t}{y_2-xt}\right)^m \left(\frac{x-y_3t}{y_3-xt}\right)^m \times \frac{xy_1t}{(x-y_1t)(y_1-xt)} \frac{xy_2t}{(x-y_2t)(y_2-xt)} \frac{xy_3t}{(x-y_3t)(y_3-xt)} . \tag{3.14}$$

Setting $\eta = 1$ picks the residues at y_1/t, y_2/t, and y_3/t, which sum to

$$\begin{aligned} Z^{\mathrm{SQED}[3]}_{g=0,B} = &- \frac{t^4y_1^2y_2}{(1-t^2)(y_1-y_2)(t^2y_1-y_2)(t^2y_1-y_3)} \\ &+ \frac{t^4y_1y_2^2}{(1-t^2)(y_1-y_2)(t^2y_2-y_1)(t^2y_2-y_3)} \\ &- \frac{t^2y_1^2y_2}{(1-t^2)(y_1-y_2)(t^2y_2-y_1)(t^2y_3-y_1)} \\ &- \frac{t^2y_1y_2^2}{(1-t^2)(y_1-y_2)(t^2y_1-y_2)(t^2y_3-y_2)} . \end{aligned} \tag{3.15}$$

To map to parameters $t' = t^{-1}$, $q_1 = y_1/y_2$, $q_2 = y_2/y_3$ in the mirror, we implement the following change of variables

$$t \mapsto t^{-1} , \tag{3.16a}$$

$$y_1 \mapsto q_1q_2y_3 , \tag{3.16b}$$

$$y_2 \mapsto q_2y_3 , \tag{3.16c}$$

which results in

$$\begin{aligned} Z^{\mathrm{SQED}[3]}_{g=0,B} =& \frac{-q_1q_2t^2((1+q_2)t^4+q_1^2q_2(1+q_2)t^4+q_1(t^4+q_2^2t^4-q_2(1+2t^2+2t^6+t^8))}{(q_1-t^2)(q_1q_2-t^2)(-q_2+t^2)(-1+q_1t^2)(-1+q_2t^2)(-1+q_1q_2t^2))} \\ =& - \frac{q_1q_2(1+q_1)t^2}{(q_1-t^2)(1-q_1t^2)} \\ &- \frac{q_1q_2^2(t^2+t^4+t^6+q_1^2(t^2+t^4+t^6)-q_1(1+t^8))}{t^2(q_1-t^2)(1-q_1t^2)} \\ &- \frac{q_1q_2^3(t^2+t^4+t^6+t^8+t^{10}+q_1^3(t^2+t^4+t^6+t^8+t^{10})-(q_1+q_1^2)(1+t^{12}))}{t^4(q_1-t^2)(1-q_1t^2)} \\ &- \frac{q_1q_2^4(t^2+t^4+...+t^{14}+q_1^4(t^2+t^4++...+t^{14})-(q_1+q_1^2+q_1^3)(1+t^{16})}{t^6(q_1-t^2)(1-q_1t^2)} \\ &+ \mathcal{O}(q_2^5) , \end{aligned} \tag{3.17}$$

where the last expression is the expansion in q_2. In this case all sectors are expected to contribute. So the computation becomes more complicated.

The contour integral formula for A-twisted index of A_2 quiver is

$$Z^{[A_2]}_{g=0,A} =(t-t^{-1})^{-2} \sum_{m_1,m_2} q_1^{m_1} q_2^{m_2} \frac{1}{(2\pi i)^2}\oint_{\mathrm{JK}} \frac{dx_1}{x_1}\frac{dx_2}{x_2} \left(\frac{x_1-y_1t}{y_1-x_1t}\right)^{m_1} \left(\frac{x_2-y_2t}{y_2-x_2t}\right)^{m_2} \left(\frac{x_1-x_2t}{x_2-x_1t}\right)^{m_1-m_2} . \tag{3.18}$$

Setting the charge vector $\eta = (1,1)$ again picks up the interior poles at

$$(y_1/t, y_2/t), (y_1 t, y_1/t^2), (y_2/t^2, y_2/t)\,.$$

However, the boundary poles involving $1/x_1$ and $1/x_2$ still need to be considered. In this case, we follow [7, (2.17)] to assign both $1/x_1$ and $1/x_2$ with the charge $(-\infty, -\infty)$. Because there is no Chern-Simons terms in $\mathcal{N}=4$ theory. It turns out that in this case none of the boundary poles are selected by the Jeffrey-Kirwan prescription as they always have a charge vector in the exact opposite direction of η.

To compare with (3.17), the contour integral formula for the B-twisted index is evaluated for $m_2 = 0,1,2,3,4$.

- The sector of $m_2 = 0$ does not have non-vanishing residues at the selected poles.
- The $m_2 = 1$ sector has contributions at $(y_1/t, y_2/t)$ and $(y_2/t^2, y_2/t)$ for $m_1 \geqslant 1$. The series of residues evaluated for each m_1 at individual poles do not resemble the expansion of rational functions. However, their sum gives

$$-\frac{q_1 q_2 (1+q_1) t^2}{(q_1 - t^2)(1 - q_1 t^2)}\,,$$

  which is identical to the q_2 term in the expansion in (3.17).
- The $m_2 = 2$ sector has contributions at $(y_1/t, y_2/t)$ and $(y_2/t^2, y_2/t)$ summing to

$$-\frac{q_1^2 q_2^2 (1+t^2)^2}{(q_1 - t^2)(1 - q_1 t^2)}$$

  for $m_1 \geqslant 2$, and contributions at $(y_1/t, y_2/t)$ and $(y_1/t, y_1/t^2)$ summing to

$$\frac{q_1^2 q_2^2 (1+t^2+t^4)}{t^2}$$

  for $m_1 = 1$. The sum of these two factors is

$$-\frac{q_1 q_2^2 (t^2 + t^4 + t^6 + q_1^2 (t^2+t^4+t^6) - q_1 (1+t^8))}{t^2 (q_1 - t^2)(1 - q_1 t^2)}\,,$$

  which is identical to the q_2^2 term in the expansion in (3.17).
- The $m_2 = 3$ sector has contributions at $(y_1/t, y_2/t)$ and $(y_2/t^2, y_2/t)$ summing to

$$-\frac{q_1^3 q_2^3 (1+t^2+t^4)(1+(1-q_1)t^2+t^4)}{t^2 (q_1 - t^2)(1 - q_1 t^2)}$$

  for $m_1 \geqslant 3$, and contributions at $(y_1/t, y_2/t)$ and $(y_1/t, y_1/t^2)$ summing to

$$\frac{q_1 q_2^3 (1+t^2+\ldots+t^8)}{t^4} + \frac{q_1^2 q_2^3 (1+t^2)^2 (1+t^4)}{t^4}$$

  for $m_1 = 1,2$. The total sum is

$$-\frac{q_1 q_2^3 (t^2+t^4+t^6+t^8+t^{10} + q_1^3 (t^2+t^4+t^6+t^8+t^{10}) - (q_1 + q_1^2)(1+t^{12}))}{t^4 (q_1 - t^2)(1 - q_1 t^2)}\,,$$

  which is identical to the q_2^3 term in the expansion in (3.17).

- The $m_2 = 4$ sector has contributions at $(y_1/t, y_2/t)$ and $(y_2/t^2, y_2/t)$ summing to

$$-\frac{q_1^4 q_2^4 (1+t^2)^2(1-q_1t^2+t^4)(1+t^4)}{t^4(q_1-t^2)(1-q_1t^2)}$$

for $m_1 \geqslant 4$, and contributions at $(y_1/t, y_2/t)$ and $(y_1/t, y_1/t^2)$ summing to

$$\frac{q_1 q_2^4(1+t^2+...+t^{12})}{t^6} + \frac{q_1^2 q_2^4(1+t^2)^2(1+t^4+t^8)}{t^6} + \frac{q_1^3 q_2^4(1+t^2+t^4)(1+t^2+...+t^8)}{t^6}$$

for $m_1 = 1, 2, 3$. The total sum is

$$-\frac{q_1 q_2^4(t^2+t^4+...+t^{14}+q_1^4(t^2+t^4++...+t^{14})-(q_1+q_1^2+q_1^3)(1+t^{16})}{t^6(q_1-t^2)(1-q_1t^2)},$$

which is identical to the q_2^4 term in the expansion in (3.17).

References

[1] K. J. Larsen, and R. Rietkerk, *MULTIVARIATERESIDUES: A Mathematica Package for Computing Multivariate Residues*, (2009). [GoodReader]

[2] N. Arkani-Hamed, F. Cachazo, C. Cheung, and J. Kaplan, *A Duality for the S Matrix*, (2009). [GoodReader]

[3] P. Griffiths, and J. Harris, *Principles of Algebraic Geometry*, Wiley, USA (1978). [GoodReader]

[4] D. Lichtblau, *Practical Computations with Groöbner Bases*, (2014). [GoodReader]

[5] L. Ferro, T. Łukowski, and M. Parisi, *Amplituhedron meets Jeffrey-Kirwan Residue*, (2018). [GoodReader]

[6] C. Closset, and H. Kim, *Comments on Twisted Indices in 3d Supersymmetric Gauge Theories*, (2016). [GoodReader]

[7] M. Bullimore, A. E. V. Ferrari, H. Kim, and G. Xu, *The Twisted Index and Topological Saddles*, (2021). [GoodReader]