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A Physicist's Introduction to Stochastic Calculus

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Abstract: This note introduces the basics of stochastic calculus building up from the Brownian motion, with the aim to understand the theory of derivative pricing in finance and the theory of quantum mechanics in physics from a unified framework of mathematics.

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1 Introduction

The notion of random processes is not an alien concept in physics, which can appear in many different phenomena ranging from hydrodynamics, quantum mechanics to particle physics. However the theory of stochastic processes is usually not part of a standard physics curriculum.

This note is primarily intended for physicists who are interested in quantitative finance. It might also be helpful to those who are familiar with stochastic processes and interested in quantum physics.

We would like to build up from first principles some working knowledge for studying stochastic processes, motivated by physical intuitions. In a typical physicist's fashion, we would try to be contempt with an appropriate level of rigour[1].

- First we to formalise some basic concepts in probability theory in Section 2;
- In Section 3 we discuss the physical phenomenon of Brownian motion to motivate the need for a modified version of calculus.
- In Section 4 we heuristically build up the basics of stochastic calculus from Brownian motion.
- In Section 5 we introduce the Black-Scholes-Merton model, which is one of the earliest attempts to formalise the pricing of financial derivatives.
- In Section 6 we discuss an alternative representation of stochastic processes in terms of path integrals.
- In Section 7, the basics of quantum mechanics is introduced, and its connection with stochastic processes is discussed.

This note is by no means a rigorous or complete attempt in addressing these topics. It is hoped that at least this note can illuminate on the fascinating connection between the theory of stochastic processes and quantum mechanics.

2 Probability

This brief section defines the basic objects in probability theory to set up notations for discussing stochastic processes.

Definition 2.1. A probability space is a triple $(\Omega, \mathcal{F}, \mathbb{P})$ where the set Ω is called the sample space, the collection $\mathcal{F} \subseteq 2^{\Omega}$ of events is a σ-algebra over Ω satisfying

- $\emptyset \in \mathcal{F}$;
- $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$;
- If $\{A_i\}$ is a countable collection of events, then $\bigcup_i A_i \in \mathcal{F}$;

and $\mathbb{P} : \mathcal{F} \to \mathbb{R}$ is a probability measure, satisfying

- $\mathbb{P}(A) \geqslant 0 \quad \forall A \in \mathcal{F}$,
- $\mathbb{P}(\Omega) = 1$,
- If $\{A_i\}$ is a countable collection of disjoint events, then $\mathbb{P}(\bigcup_i A_i) = \sum_i \mathbb{P}(A_i)$.

[1]In practice this often means as little rigour as we can get away with.

Definition 2.2. A random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a map $X : \Omega \to \mathbb{R}$ such that $\{\omega \mid X(\omega) \leqslant x\} \in \mathcal{F}$ for all $x \in \mathbb{R}$.

Definition 2.3. A stochastic process is a collection of random variables $\{X(t)\}_{t \in T}$ where T is some index set, which can be equivalently understood as a distribution over paths.

3 Brownian Motion

Brownian motion describes the random motion of particles suspended in some medium. It is a stochastic process in physics that naturally gives arise to the normal distribution. The works by Einstein [2] and Langevin [3] ultimately lead to the development of modern theory of stochastic processes.

Consider a one-dimensional Brownian motion in space x and time t. After a short time interval τ, the position of a particle is randomised, which may be described by a random variable $X(t)$. A particle moves from x to $x + z$ in time τ, where the displacement $Z(\tau) = X(t + \tau) - X(t)$ is an independent random variable with some probability density $\phi(z)$. This probability density depends on the intrinsic properties of the particles and the medium.

Physically the number density ρ at time $t + \tau$ can be computed from this probability density ϕ as

$$\rho(x, t + \tau) \equiv \int_{-\infty}^{\infty} dz\, \phi(z) \rho(x + z, t)\,, \tag{3.1}$$

where the right hand side is the expected number density $\mathbb{E}_z(\rho(x + z, t))$ by definition. Note that this relation states that the time evolution of $\rho(x, t)$ only depends on the status quo at time t, but not information from the past. Therefore this is actually a form of the Markov property.

This Markov property (3.1) can also be justified mathematically [1] by writing

$$\begin{aligned} \rho(x, t + \tau)dx &= \mathbb{P}(\{x \leqslant X(t + \tau) < x + dx\}) \\ &= \int_{-\infty}^{\infty} dy\, \mathbb{P}(\{x \leqslant X(t + \tau) < x + dx \mid X(t) = y\}) \rho(y, t)\,, \end{aligned}$$

where the conditional probability in the integrand is $\phi(x - y)dx$.

For a single particle, the number density $\rho(x, t)$ simply becomes the probability density function of the position $X(t)$ of the particle.

The probability density $\rho(x, t)$ of $X(t)$ can be expanded as

$$\rho(x + z, t) = \rho(x, t) + z\partial_x \rho(x, t) + \frac{1}{2} z^2 \partial_x^2 \rho(x, t) + \cdots\,, \tag{3.2a}$$

$$\rho(x, t + \tau) = \rho(x, t) + \tau\partial_t \rho(x, t) + \cdots\,. \tag{3.2b}$$

When evaluated in the integral in (3.1), the first order term in the spatial expansion (3.2a) is proportional to the first moment $\mathbb{E}(Z)$, which vanishes since $\phi(z)$ is even due to the reflection symmetry of the system.

Therefore substituting in the leading order terms from (3.2a) and (3.2b) into (3.1) gives a diffusion equation

$$\frac{\partial \rho}{\partial t} = \frac{1}{2} D \frac{\partial^2 \rho}{\partial x^2}\,, \tag{3.3a}$$

where the diffusion coefficient D is proportional to the second moment $\mathbb{E}(Z^2)$ via

$$D = \frac{1}{\tau} \int_{-\infty}^{\infty} dz\, \phi(z) z^2\,. \tag{3.3b}$$

Instead of describing the trajectory $x(t)$ of the Brownian particle directly, the diffusion equation governs the evolution of its probability distribution of all possible trajectories. This is not dissimilar to quantum mechanics where the wave function, roughly the "square root" of probabilities is governed by the Schrödinger's equation.

For a Brownian particle starting from $X(t = 0) = 0$, i.e., with a boundary condition $\rho(x, 0) = \delta(x)$, the diffusion equation has the solution

$$\rho(x,t) = \frac{1}{\sqrt{2\pi Dt}} \exp\left(-\frac{x^2}{2Dt}\right), \tag{3.4}$$

which is a normal distribution $\mathcal{N}(\mu = 0, \sigma^2 = Dt)$. Note that the spread of the particles is proportional to $\sqrt{t}$, which is a signature for stochastic calculus. Due to the Markov property (3.1) of the Brownian motion, any increment can be described by the same distribution, i.e.,

$$X(t) - X(s) \sim \mathcal{N}(0, t-s)\,. \tag{3.5}$$

These properties can abstracted to formally define [4] the Brownian motion.

Theorem 3.1 (Brownian Motion)**.** There exists a probability distribution over the set of continuous functions $B : \mathbb{R}_{\geqslant 0} \to \mathbb{R}$ such that

- The functions start from the origin, i.e.,

$$\mathbb{P}(B(0) = 0) = 1\,.$$

- For all $0 \geqslant s < t$, the increment is stationary, i.e.,

$$B(t) - B(s) \sim \mathcal{N}(0, t-s)\,.$$

- The increments $B(t_i) - B(s_i)$ are independent for all non-overlapping intervals $\{[s_i, t_i]\}$.

This distribution is called the Brownian motion. Then a particular instance of the continuous function chosen according to the Brownian motion is referred to as a sample Brownian path.

The Brownian motion can be formally obtained as the limit of the simple random walk. This is in agreement with our physical intuitions. When observing the movement of the particle suspended in a medium, its position is recorded in discrete time. The particle is bombarded by a large number of independent collisions from the medium molecules, causing its recorded position to "jump" randomly in a discrete manner. The actual physical process is the underlying continuous process which should reveal as we measure in increasingly finer time intervals.

The Brownian motion plays a special role in stochastic processes, in analogy to the role of normal distributions in statistics due to the central limit theorem. Due to its ubiquitous usefulness in modelling a wide range of phenomena in physics and finance, we would like to be able to do analysis on trajectories of the Brownian motion. Unfortunately the standard calculus fails, roughly due to the fact that a sample Brownian path just jumps too much no matter how much we zoom into it.

4 Stochastic Calculus

4.1 Motivation

We would like to devise a modification of calculus that can be applied to stochastic processes. The hint towards a successful modification can be heuristically derived by considering "variations" of the Brownian motion.

4.1.1 Linear Variation

To make the statement of non-differentiability of the Brownian motion precise, consider the first order variation $B(t+\epsilon) - B(t)$.

Proposition 4.1. The Brownian motion $B(t)$ is not differentiable at all $t > 0$.

Proof. Suppose $B(t)$ is differentiable with a derivative $dB(t) = A dt$ for some finite A. By the mean value theorem the infinitesimal variation is given by

$$B(t+\tau) - B(t) = \tau A\,. \tag{4.1}$$

Then the inequality

$$|B(t+\epsilon) - B(t)| \leqslant \tau |A| \tag{4.2}$$

holds for all $\epsilon < \tau$, which can be described in terms of the maximum process

$$M(\tau) \coloneqq \max_{\epsilon<\tau}\{B(\epsilon)\} \tag{4.3}$$

as

$$|M(\tau)| \leqslant \tau |A|\,. \tag{4.4}$$

We would like to show that the probability $\mathbb{P}(|M(\tau)| \leqslant \tau|A|)$ is zero.

Since the Brownian motion is symmetric with respect to reflections, the probabilities for its trajectory to deflect either ways must be the same, i.e.,

$$\mathbb{P}(B(t) - B(s) \geqslant 0) = \mathbb{P}(B(t) - B(s) < 0) \qquad \forall \quad 0 < s < t\,. \tag{4.5}$$

Let τ_a be the stopping time

$$\tau_a = \min_{B(s)=a}\{s\}$$

when the Brownian motion reaches a for the first time. The reflection symmetry

$$\mathbb{P}(B(t) - B(\tau_a) \geqslant 0 \mid \tau_a < t) = \mathbb{P}(B(t) - B(\tau_a) < 0 \mid \tau_a < t) \tag{4.6}$$

still holds when conditioned, due to the so-called strong Markov property. This seemingly arbitrary construction allows us to re-write the reflection principle as

$$\mathbb{P}(B(t) \geqslant a \mid \tau_a < t) = \mathbb{P}(B(t) < a \mid \tau_a < t)\,, \tag{4.7}$$

which can be used to evaluate

$$\begin{aligned}
\mathbb{P}(M(t) > a) &= \mathbb{P}(\tau_a < t) \\
&= \mathbb{P}(\{B(t) - B(\tau_a) \geqslant 0\} \cap \{\tau_a < t\}) + \mathbb{P}(\{B(t) - B(\tau_a) < 0\} \cap \{\tau_a < t\}) \\
&= 2\mathbb{P}(B(t) > a \mid \tau_a < t)\mathbb{P}(\tau_a < t) \\
&= 2\mathbb{P}(B(t) > a)\,.
\end{aligned} \tag{4.8}$$

This property can then be used to compute

$$\lim_{\tau\to 0} \mathbb{P}(M(\tau) > \tau|A|) = 2\mathbb{P}(B(\tau) > \tau|A|) = 1\,, \tag{4.9}$$

Therefore this leads to the conclusion

$$\mathbb{P}(|M(\tau)| \leqslant \tau|A|) = 0\,, \tag{4.10}$$

which means $B(t)$ is not differentiable with probability one. □

4.1.2 Quadratic Variation

The non-differentiability comes from the fact that a Brownian path jumps so much that even taking the infinitesimal limit does not make it smooth enough for differentiation. This statement can be made precise by considering the quadratic variation $(B(t+\epsilon) - B(t))^2$.

Theorem 4.2. For a partition $\Pi = \{t_i\}_0^n$ of the interval $[0, T]$, the Brownian motion $B(t)$ satisfies

$$\lim_{n\to\infty} \sum_{i=1}^{n} (B(t_i) - B(t_{i-1}))^2 = T\,, \tag{4.11}$$

which is equivalent to the infinitesimal expression

$$dB^2 = dt\,. \tag{4.12}$$

Proof. Without loss of generality, consider the uniform partition $t_i = iT/n$. By definition of the Brownian motion, the increments are normally distributed independent random variables

$$Z_i := (B(t_i) - B(t_{i-1})) \sim \mathcal{N}(0, T/n)\,. \tag{4.13}$$

The summation becomes

$$\sum_{i=1}^{n} (B(t_i) - B(t_{i-1}))^2 = \sum_{i=1}^{n} Z_i^2\,, \tag{4.14}$$

which is simply over a large number of samples of Z_i. By the strong law of large numbers, the limit is given by

$$\lim_{n\to\infty} \sum_{i=1}^{n} (B(t_i) - B(t_{i-1}))^2 = n\mathbb{E}(Z_i^2) = T\,. \tag{4.15}$$

□

Note that for a smooth function $f(t)$, this quadratic variation vanishes due to the mean value theorem, which is why the standard calculus only considers up to the first order variations. This non-vanishing quadratic variation of the Brownian motion turns out to be the key ingredient to devise a working formalism of calculus on stochastic processes.

4.2 Itô Differential

Consider a smooth function $f(x)$. The infinitesimal differential element of $f(x)$ can be written as the Taylor series

$$df = \frac{df}{dx}dx + \frac{1}{2}\frac{d^2f}{dx^2}dx^2 + \cdots\,. \tag{4.16}$$

The non-leading variations vanish if x an ordinary variable. However if f is a function of $x(t)$ as a sample path of the Brownian motion $B(t)$, then the quadratic term must be retained

$$df = \frac{df}{dx}dB + \frac{1}{2}\frac{d^2f}{dx^2}dB^2 = \frac{df}{dx}dB + \frac{1}{2}\frac{d^2f}{dx^2}dt\,, \tag{4.17}$$

where the identity (4.12) has been used for the second equality. The stochastic nature of $x(t)$ gives the differential df an explicit dependency on time t. This is in stark contrast with ordinary variables where the dependency on time would have been exclusively via the variation dx.

The Brownian motion can be generalised with the addition of a "drift" term in addition to the diffusion.

Definition 4.1. A stochastic process $X(t)$ satisfying

$$dX = \mu dt + \sigma dB \tag{4.18}$$

is called the Itô process, where $B(t)$ is the Brownian motion.

The above function differential can then be generalised with respect to the Itô process.

Theorem 4.3 (Itô Lemma)**.** Let $f(t, x)$ be a smooth function, where x is a sample path of the Itô process $X(t)$. The differential is then given by

$$df = \left(\frac{\partial f}{\partial t} + \mu \frac{\partial f}{\partial X} + \frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial X^2}\right) dt + \sigma \frac{\partial f}{\partial X} dB \,. \tag{4.19}$$

4.3 Itô Integral

The integration of stochastic processes can be defined as the inverse of differentiation.

Definition 4.2. Let $f(t, x)$ and $g(t, x)$ be smooth functions, where $x(t)$ is a sample path of the Brownian motion $B(t)$. The integration of f and g is defined by

$$F(t, x) := \int dB\, f(t, B) + \int dt\, g(t, B) \tag{4.20}$$

if and only if

$$dF = f(t, B) dB + g(t, B) dt \,. \tag{4.21}$$

5 Black-Scholes-Merton Model

Suppose that the price $S(t, B)$ of a stock can be modelled [5] by a log-normal distribution with Brownian noise

$$\frac{dS}{S} = \mu dt + \sigma dB \,. \tag{5.1}$$

Then the price $f(t, S)$ of some derivative contingent on S must obey

$$df = \left(\frac{\partial f}{\partial t} + \mu S \frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2}\right) dt + \sigma S \frac{\partial f}{\partial S} dB \,. \tag{5.2}$$

according to the Itô lemma. Note that the only "risk" comes from the non-deterministic Brownian motion dB.

We would like to construct a portfolio of the stock and its derivative such that this risk is eliminated. Consider a "delta-hedge" portfolio consisting of one derivative in short position and $\frac{\partial f}{\partial S}$ stocks in long position, which has value given by

$$\Pi = -f + \frac{\partial f}{\partial S} S \,. \tag{5.3}$$

The change in the value of the portfolio from τ to $\tau + \Delta\tau$ is then

$$\Delta\Pi = \int_{\tau}^{\tau+\Delta\tau} \left(-df + \frac{\partial f}{\partial S} dS\right) = \int_{\tau}^{\tau+\Delta\tau} dt \left(-\frac{\partial f}{\partial t} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2}\right) . \tag{5.4}$$

In the last equality the non-deterministic terms involving dB from the stock price and the derivative price are cancelled out exactly, which is the desired "risk-free" property for the portfolio. Therefore the portfolio value must change at exactly the same rate as any other risk-free assets such that no arbitrage can occur, i.e.,

$$\Delta\Pi = \int_{\tau}^{\tau+\Delta\tau} dt\, r\Pi\,, \tag{5.5}$$

where r is the risk-free interest rate.

Equating (5.4) and (5.5) gives us a differential equation

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf\,, \tag{5.6}$$

which is called the Black-Scholes-Merton equation.

6 Feynman-Kac Formula

There exists a different interpretation of stochastic processes in terms of path integrals [6]. For a discrete sample path $\Gamma = \{x_0, \dots, x_N\}$ starting at some fixed x_0, the probability for this exact path to occur is given by

$$\mathbb{P}(\Gamma) = \prod_{i=1}^{N} P(x_i - x_{i-1})\,, \tag{6.1}$$

where the increment distribution $P(\delta x)$ is assumed to be independent and identical. Consider the path-dependent quantity

$$\mathcal{O} = \exp \sum_{i=1}^{N} V(x_i)\,, \tag{6.2}$$

where $V(x)$ is some function of x. We are interested in its average over all possible paths starting at x_0, which can be written as

$$\langle \mathcal{O} \rangle_N = \int \prod_{i=1}^{N} dx_i\, P(x_i - x_{i-1}) \exp V(x_i)\,. \tag{6.3}$$

This is a discrete version of a path integral which is ubiquitous in the study of quantum physics.

Imposing the boundary condition $x_N = z$ allows us to restrict our attention to the average over only the paths ending at some fixed z,

$$f(z, N) := \int \prod_{i=1}^{N} dx_i\, P(x_i - x_{i-1}) \exp V(x_i) \delta(x_N - z)\,, \tag{6.4}$$

which simply evaluates to

$$\exp V(z) \int \prod_{i=1}^{N-1} dx_i\, P(x_i - x_{i-1}) \exp V(x_i) P(x_N - x_{N-1})\,.$$

The average over all possible paths can be recovered as

$$\langle \mathcal{O} \rangle_N = \int dz\, f(z, N) \tag{6.5}$$

using the definition of the Dirac delta distribution.

This seemingly arbitrary artificial construction $f(z, N)$ turns out to not only provide a powerful tool for stochastic simulations, but also connects with the world of quantum mechanics. In particular we would like to show that $f(z, N)$ obeys a form of differential equation which can be interpreted as a "proliferating" Brownian motion.

Consider moving one step further in the path. The restricted path integral becomes

$$\begin{aligned} f(z, N+1) &= \int \prod_{i=1}^{N+1} dx_i \, P(x_i - x_{i-1}) \exp V(x_i) \delta(x_{N+1} - z) \\ &= \int dx_{N+1} \, \delta(x_{N+1} - z) e^{V(x_{N+1})} \int \prod_{i=1}^{N} dx_i \, P(x_i - x_{i-1}) e^{V(x_i)} P(x_{N+1} - x_N) \\ &= e^{V(z)} \int \prod_{i=1}^{N} dx_i \, P(x_i - x_{i-1}) e^{V(x_i)} P(z - x_N) \, . \end{aligned} \tag{6.6}$$

Substituting in

$$P(z - x_N) = \int dy \, P(z - y) \delta(x_N - y) \tag{6.7}$$

gives us a recursive relation

$$f(z, N+1) = e^{V(z)} \int dy \, P(z - y) f(y, N) \, , \tag{6.8}$$

which makes sense physically as the path integral transitions one step further into the next state.

To derive some differential equation for f, we need to take the continuous limit, where the path increment becomes $z - y = \epsilon$ for some infinitesimal ϵ during an infinitesimal time evolution $t \mapsto t + \delta t$. For $\mathcal{O}$ to be a well-behaved function, the function V must be of order δt, i.e.,

$$V(x) = U(x) \delta t \, . \tag{6.9}$$

Then the recursive relation becomes

$$\begin{aligned} f(z, N+1) &= e^{V(z)} \int d\epsilon \, P(\epsilon) f(z - \epsilon, N) \\ &= (1 + U(z)\delta t + \cdots) \int d\epsilon \, P(\epsilon) \left(f(x, N) - \epsilon \frac{\partial f(x, N)}{\partial x} + \frac{1}{2} \epsilon^2 \frac{\partial^2 f(x, N)}{\partial x^2} + \cdots \right) \bigg|_{x=z} \, . \end{aligned} \tag{6.10}$$

Assuming $P(\epsilon)$ has mean $\mu\delta t$ and variance $\sigma^2 \delta t$, with vanishing higher order moments, the equation reduces to

$$f(z, T + \delta t) - f(z, T) = -\mu \frac{\partial f(x, T)}{\partial x} \bigg|_{x=z} \delta t + \frac{1}{2} \sigma^2 \frac{\partial^2 f(x, T)}{\partial x^2} \bigg|_{x=z} \delta t + U(z, T) \delta t \tag{6.11}$$

up to linear terms in δt. Therefore the time derivative

$$\frac{\partial f(x, T)}{\partial t} \coloneqq \lim_{\delta t \to 0} \frac{f(x, T + \delta t) - f(x, T)}{\delta t} \tag{6.12}$$

of $f(x, t)$ at $t = T$ satisfies

$$\frac{\partial f(x, T)}{\partial t} = U(x) f(x, T) - \mu \frac{\partial f(x, T)}{\partial x} + \frac{1}{2} \sigma^2 \frac{\partial^2 f(x, T)}{\partial x^2} \, , \tag{6.13}$$

which is a diffusion equation with a source term $U(x) f(x, T)$ and a drift term $-\mu \frac{\partial f(x,T)}{\partial x}$. The function $f(x, t)$ can be interpreted as the density of Brownian particles which which proliferate at rate $U(x)$.

Assuming P is a gaussian distribution $\mathcal{N}(\mu dt, \sigma\sqrt{dt})$, the integrand in (6.3) becomes

$$\lim_{\delta t\to 0}\prod_{i=1}^{N}\exp\left(-\frac{(x_i - x_{i-1} - \mu\delta t)^2}{2\sigma^2\delta t} + U(x_i)\delta t\right) = \exp\int_0^T dt\left(-\frac{1}{2\sigma^2}\left(\frac{dx}{dt} - \mu\right)^2 + U(x)\right) \quad (6.14)$$

in the continuous time limit. The function $f(z, T)$ can then be written as a formal integral over all possible paths as

$$f(z, T) = \int_{x\in\hat{\Gamma}} \mathcal{D}x \,\exp\int_0^T dt\left(-\frac{1}{2\sigma^2}\left(\frac{dx}{dt} - \mu\right)^2 + U(x)\right), \quad (6.15)$$

where $\mathcal{D}x$ is the integration measure over all paths, and $\hat{\Gamma} = \{x(t) \mid x(0) = x_0), x(T) = z\}$ is the set of all paths starting at x_0 and ending at z. It is a solution to the differential equation (6.13). This is called the Feynman-Kac path integral interpretation, which is a not only a conceptually beautiful construction, but also provides a practical formula for simulations using Monte-Carlo methods.

7 Quantum Mechanics

There exists two mainstream formulations for quantum physics: the operator formalism and the path integral formalism, which are equivalent and complementary. Traditionally quantum mechanics is introduced in the operator formalism, while path integrals are only established for dealing with quantum field theories.

In the operator formalism, the probability density to observe a particle at (x, t) is given by the square of a "wave function" $\psi(x, t)$, which obeys the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\psi(x, t) = \hat{H}\psi(x, t)\,. \quad (7.1)$$

The differential operator $\hat{H}$ is called the hamiltonian

$$\hat{H} := -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\,, \quad (7.2)$$

expressed in "position representation", which measures the energy of the wave function $\psi(x, t)$. The term $-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}$ gives the kinetic energy of the particle, while $V(x)$ is the potential energy stored in the system.

The Schrödinger equation has the same form as the "drift-diffusion-proliferation" equation (6.13) with

$$\mu = 0\,, \quad \sigma^2 = \frac{i\hbar}{m}\,, \quad U(x) = -\frac{i}{\hbar}V(x)\,. \quad (7.3)$$

There is no "drift" any more, while strikingly the variance has become imaginary. The path integral (6.15) for the wave function is then

$$\psi(z, T) = \int_{x\in\hat{\Gamma}} \mathcal{D}x \,\exp\left[\frac{i}{\hbar}\int_0^T dt\left(\frac{1}{2}m\left(\frac{dx}{dt}\right)^2 - V(x)\right)\right], \quad (7.4)$$

where the integration is over the set $\hat{\Gamma}$ of all paths starting at some fixed $x(0) = x_0$ and ending at $x(T) = z$. The complex phase in the integrand is called the action

$$S := \frac{1}{\hbar}\int_0^T dt\, L\,, \quad (7.5)$$

where L is the lagrangian

$$L \coloneqq \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 - V(x)\,. \tag{7.6}$$

This is the path integral formalism of quantum mechanics. In the classical limit $\hbar \to 0$, we expect that only the deterministic classical path contributes to the integral, which is given by the Euler-Lagrangian equation

$$0 = \frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} \tag{7.7}$$

derived $\delta S = 0$ from the principle of least action, which is equivalent to the Newton's second law. The quantum effects add contributions from fuzzy paths into path integral in addition to the classical path.

The interpretation is that a quantum particle must travel through all possible paths. Taking the double slit experiment as an example, an electron is fired at x_0 at the beam gun, landing at some z on the screen behind the double slit after time T. In the Dirac notation, the initial and final states are denoted as $|x_0\rangle$ and $|z\rangle$. We would like to compute the "transition amplitude" between these two states after a time evolution of T given by

$$\psi(z, T) \equiv \langle z| e^{-\frac{i}{\hbar}\int_0^T dt\,\hat{H}} |x_0\rangle\,, \tag{7.8}$$

where the hamiltonian $\hat{H}$ describes the operation of the double slit on the electron. It can be shown [7] that the transition amplitude can be written as the path integral (7.4).

The technique of path integrals is arguably the most important tool in the study of modern quantum physics. The Feynmann diagrams representing various fundamental particle interactions are actually book-keeping devices for computing these path integrals perturbatively. It underlies the development of the standard model of particle physics, which is the most successful theory in terms of its empirical predictions.

However, the construction of the path integral is not mathematically well-defined. The problem superficially lies in the imaginary exponent in the integrand mandates, which makes the convergence of the integral a rather subtle problem [8]. In a typical physicist's fashion, we can avoid this difficulty entirely via the so-called Wick rotation

$$t \mapsto -i\tau\,, \tag{7.9}$$

which "euclideanises" the lorentizian spacetime. The path integral for the euclidian quantum mechanics is then given by

$$\psi(z, T) = \int_{x\in\hat{\Gamma}} \mathcal{D}x\, \exp\left[-\frac{1}{\hbar}\int_0^T d\tau \left(\frac{1}{2}m\left(\frac{dx}{d\tau}\right)^2 + V(x)\right)\right]. \tag{7.10}$$

Note that this is not the same theory as the standard quantum mechanics. But it has been proven to be a very useful toy model to help us understand the real quantum theory.

We have shown that the wave function of quantum mechanics obeys the same mathematics as the Brownian motion. However we need to be clear that quantum particles are not Brownian particles. It is not their probability distributions that follow the same differential equations as Brownian motions, but their wave functions, which are the "square root" of probability distributions. Nevertheless this is quite remarkable that two processes in very different settings turn out to behave according to the same mathematics.

There are also other attempts to make connections between quantum physics. For example, the Hamilton-Jacobi formulation of classical physics can be shown [9] to be "stochasticised" to re-produce quantum mechanics. However the generalisation to field theories has not been well understood.

8 Discussion

We have demonstrated the striking connection that the mathematics underlying financial derivatives and quantum mechanics is actually the same. However this should be expected considering their shared non-deterministic natures.

Since the two fields are largely developed independently with a wealth of knowledge in their own domain, this connection is expected to open up a lot of interactions between the two fields. At least it would be interesting to be able to apply techniques from one field to another.

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